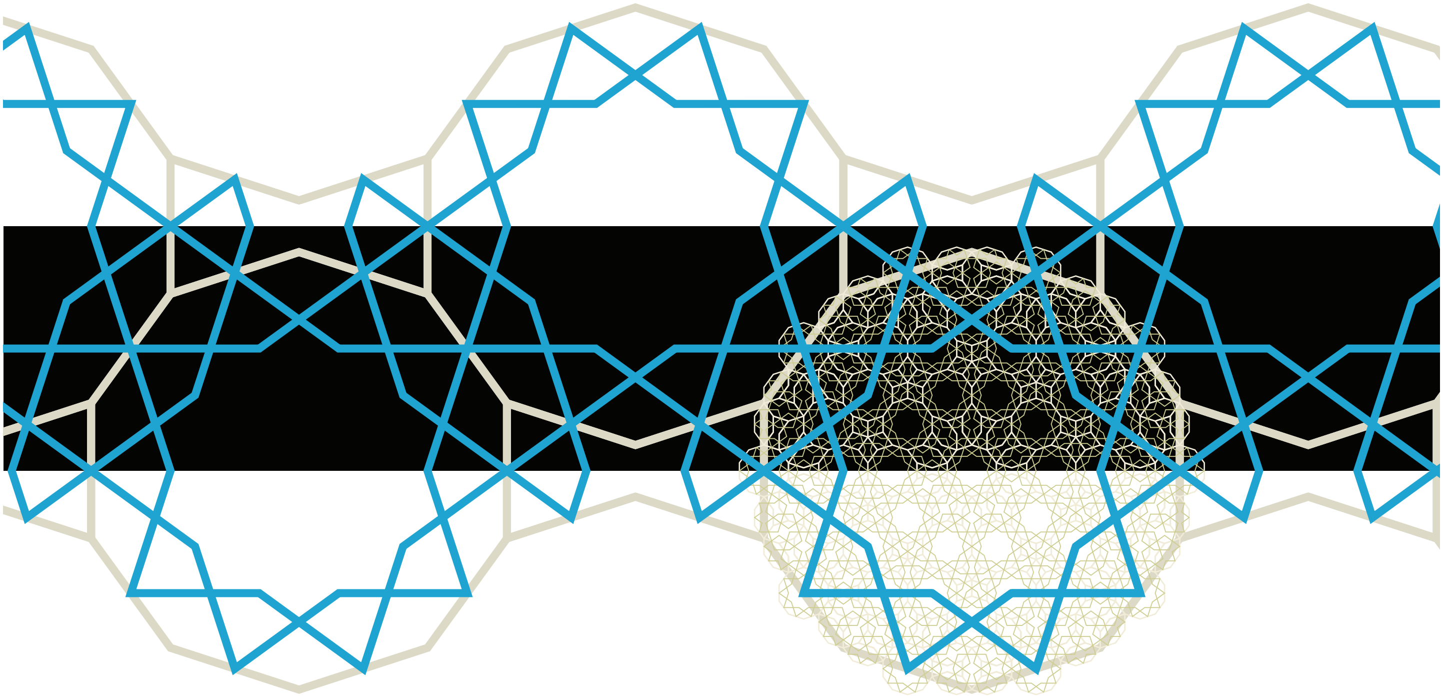
Quasiperiodic Tilings

Byera Hadley Travelling Scholarship Report

Andrew Willes, University of Technology, Sydney



2 Periodic and aperiodic tilings

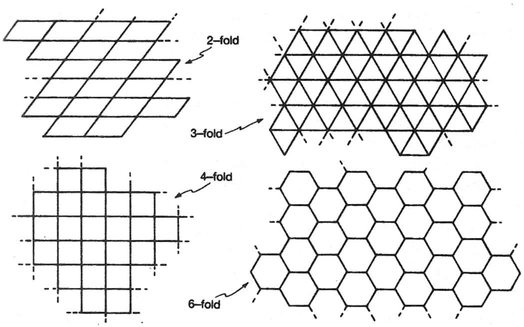
The mathematics of tiling is concerned with the problem of surfacing the plane without gaps or overlaps, using a finite set of tile shapes. This can be achieved with a single tile, and certain tile shapes, such as equilateral triangles, squares and regular hexagons, generate a periodic tiling, where a copy of the tiling on transparent paper can be translated to a new position that exactly overlays the original tiling. More complex shapes can also produce a periodic tiling, such as M.C. Escher's tilings.

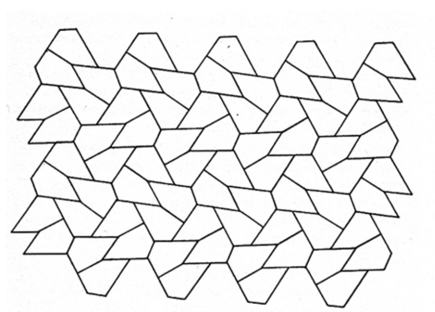
However it is not possible to tile the plane with regular pentagons. A more general result is that periodic tiling is only possible for tiles with 2-fold, 3-fold, 4-fold or 6-fold rotational symmetry[1]. For example the periodic tiling of irregular pentagons shown below[2] has 4-fold rotational symmetry; or in other words, the tiling is reproduced by rotating the entire pattern about each centre of symmetry by 360°/4.

In the 1970s a set of two tiles was discovered that can tile the plane aperiodically, but never periodically. The Penrose tiling belongs to a special subclass of aperiodic tilings, known as a quasiperiodic tiling - it appears to be nearly periodic, and almost possesses the 5-fold symmetry which is forbidden in periodic tilings.

Current scientific interest in quasiperiodic tilings relate to their correspondence with the atomic structure of solids. Until the 1980s, all known atomic lattices possessed either 2-,3-, 4- or 6-fold rotational symmetry. Metal alloys, such as aluminium-manganese, have since been discovered which appear to break the accepted geometric rules of atomic packing, with an atomic structure with icosahedral symmetry that is the three-dimensional analogue of the the two-dimensional quasiperiodic tilings. Their quasicrystalline structure lies somewhere between the crystalline order of diamonds and the amorphous disorder of glass.

There is a diverse variety of pentagonal and decagonal tiling patterns in medieval Islamic architecture. The properties of one such tiling from the Timurid-era Darb-e-Imam Shrine (1453) in Esfahan, Iran, which is self-similar and quasiperiodic, was recently published in Science[3]. Although the authors are cautious in their conclusions, this finding suggests that quasiperiodic tilings were known to Persian

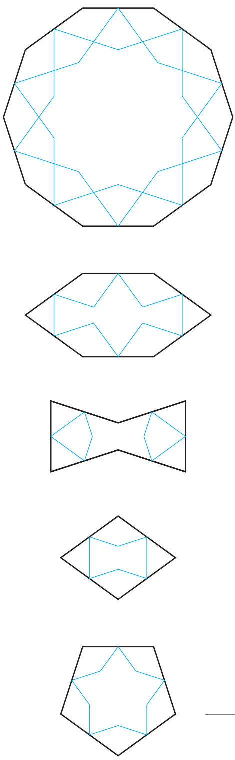




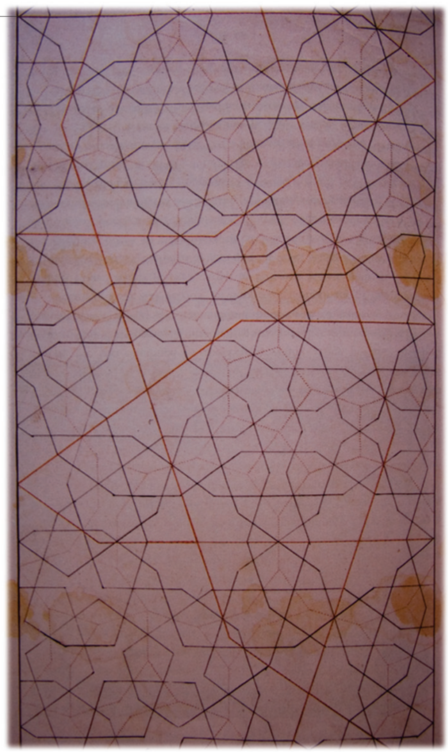
1 Figure from Penrose, R., 1989, The Emperor's New Mind, Oxford: Oxford University Press, p. 563.
2 Penrose [1989], p. 172.

3 Lu, P.J. & Steinhardt, P.J., 2007, Decagonal and Quasi-Crystalline Tilings in Medieval Islamic Architecture, Science, 315, 1106-1110.

Tiling detail: Darb-e-Imam Shrine, Esfahan.



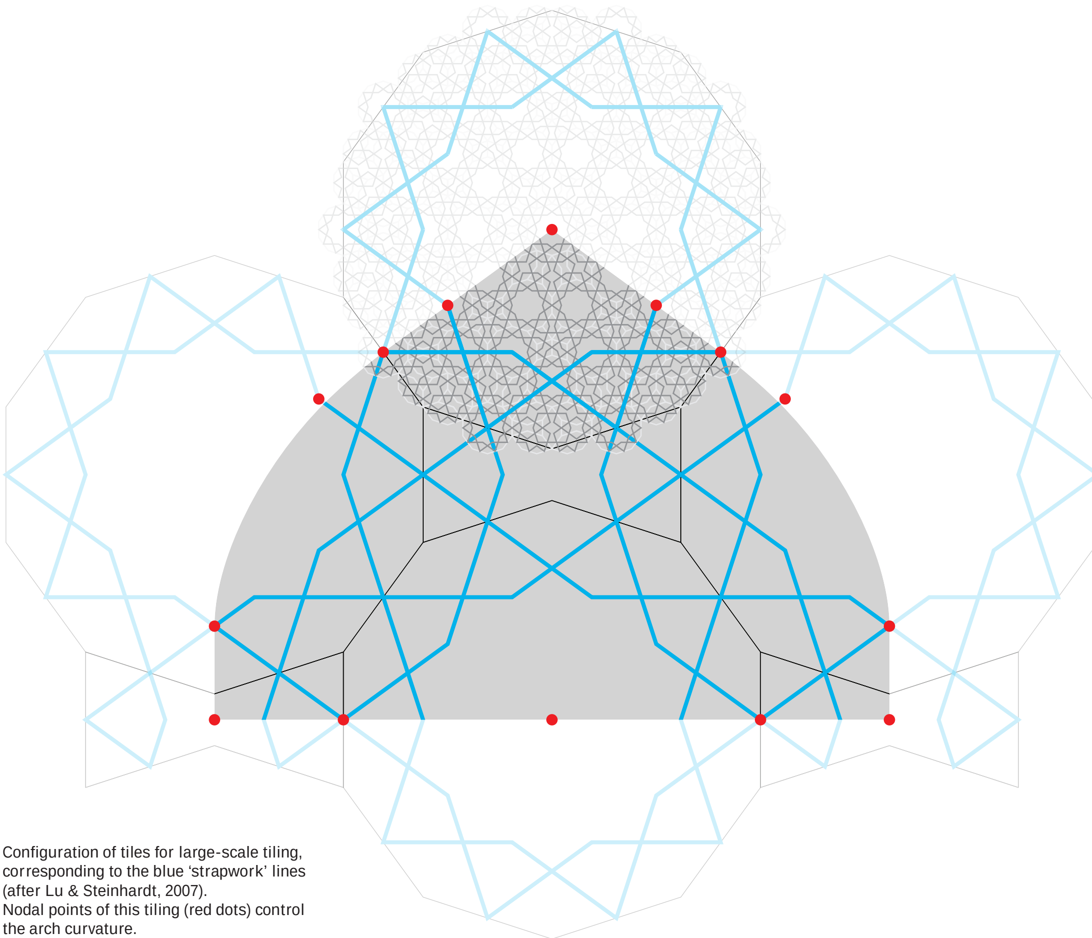
Set of five girih tiles.



Girih tile outlines (dashed red lines) on Topkapi scroll.

4

Tiling detail: Darb-e-Imam Shrine, Esfahan.



Configuration of tiles for large-scale tiling, corresponding to the blue ‘strapwork’ lines (after Lu & Steinhardt, 2007).
Nodal points of this tiling (red dots) control the arch curvature.

scientists and architects five hundred years prior to their western counterparts.

The girih tile theory[4] postulates that Islamic tilings with pentagonal symmetry were constructed, both conceptually and in the building process, using a basis system of five template or girih tiles[5]. The girih tile framework alleviates the difficulties associated with traditional techniques of tile patterning using a ruler and compass, where slight errors in angle would propagate and be magnified over the thousand of lines that comprise the most elaborate tilings.

The strongest evidence for the girih-tile method is in a Timurid-era scroll, now in the Topkapi Palace, Istanbul, where the outlines of the girih tiles are outlined in thin dotted red lines[6]. The outlines of the girih tiles are not evident in the final tiling – they are purely a design guide. Also evident in the Topkapi scroll are thick red lines, which correspond to the larger scale tiling which exploit the self-similarity properties of the girih tiles.

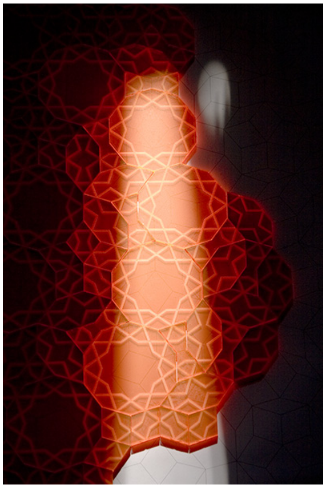
4 Lu & Steinhardt [2007].
5 girih means ‘knot’ in Farsi.
6 Scroll image from Lu & Steinhardt [2007].

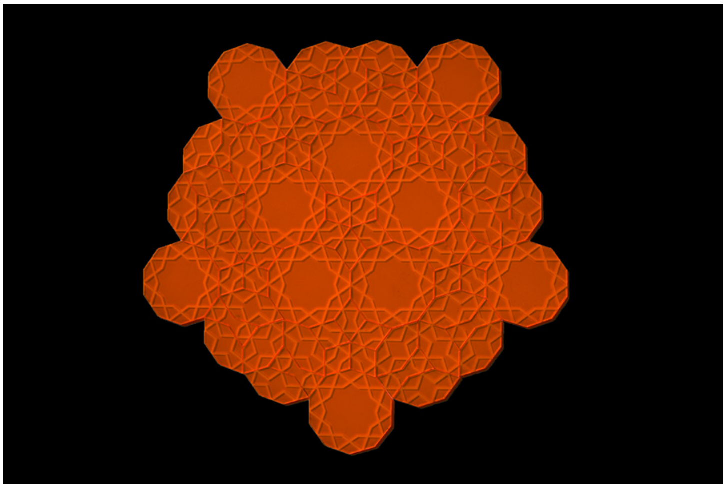
The tiling is self-similar in the sense that each girih meta-tile can be decomposed into an arrangement of girih tiles at a smaller scale, corresponding to the individual tiling scale. The decomposition of another Darb-e-Imam tiling into the two scales of girih tiles is illustrated on page 4, where the large-scale (meta) tiling determines the positions of the blue lines connecting the yellow stars, and the small scale tiling corresponds to the scale of individual tiles.

There is also a dual relationship between the tiling geometry (at the meta-tiling scale) and the structural geometry, where nodal points of the meta-scale girih tiles constrain the arch geometry in a similar manner to the control points of Bézier curves.

The girih tiles are directly related to Penrose tiles[7]. One of the Darb-e-Imam tilings has a one-to-one correspondence with a Penrose tiling[8]. The photos below display laser-cut acrylic girih tiles arranged in correspondence with a Penrose tiling.

7 Makovicky, E., 1992, in Fivefold Symmetry, Ed. I. Hargittai, Singapore: World Scientific, pp.67-86.
8 Lu & Steinhardt [2007].





6 Friday Mosque, Esfahan

The Friday Mosque in Esfahan displays numerous historical layers accreted over many centuries. The first mosque on the site of the Friday Mosque was constructed during the Abbasid period (8th century). Later Abassid mosque structures from the 9th century corrected the misalignment of the Qibla wall of the original mosque. With further additions during the Seljuk period (11th – 12th century), the Friday Mosque evolved into a four-iwan structure, which was to become the standard model for subsequent Persian mosques[1].

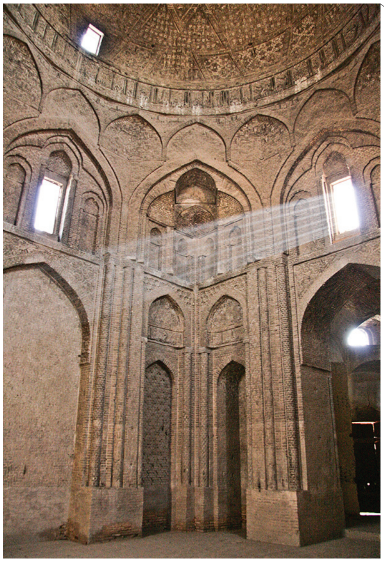
Each of the four iwans underwent numerous alterations in later periods. One example is the Safavid-era muqarnas system in the ceiling of the East Iwan (late 16th century) which was the third such system in this location.The decoration of the West Iwan, which is one major focus in this project, is a mixture of Timurid-era (15th century) and Safavid-era (16th – early 18th century) tiling.

There is an embedded parametric relationship between the structure (the arc length of the arch) and the tiling decoration in the West Iwan of the Friday Mosque. Nine unit cells of the meta-tiling (the blue 'strapwork' lines) extend along the length of the arch (pages 10-11). This particular tiling is periodic at the meta-tiling scale and is self-similar. The self-similarity is related to the self-similarity of Penrose tiles discussed later in this report. In this instance, the parametric relationship between tiling and building form is complementary; i.e., to accommodate a longer arch length without adding a whole additional unit cell, the meta-tiling scale could be increased slightly, such that the arch width increases proportionately.

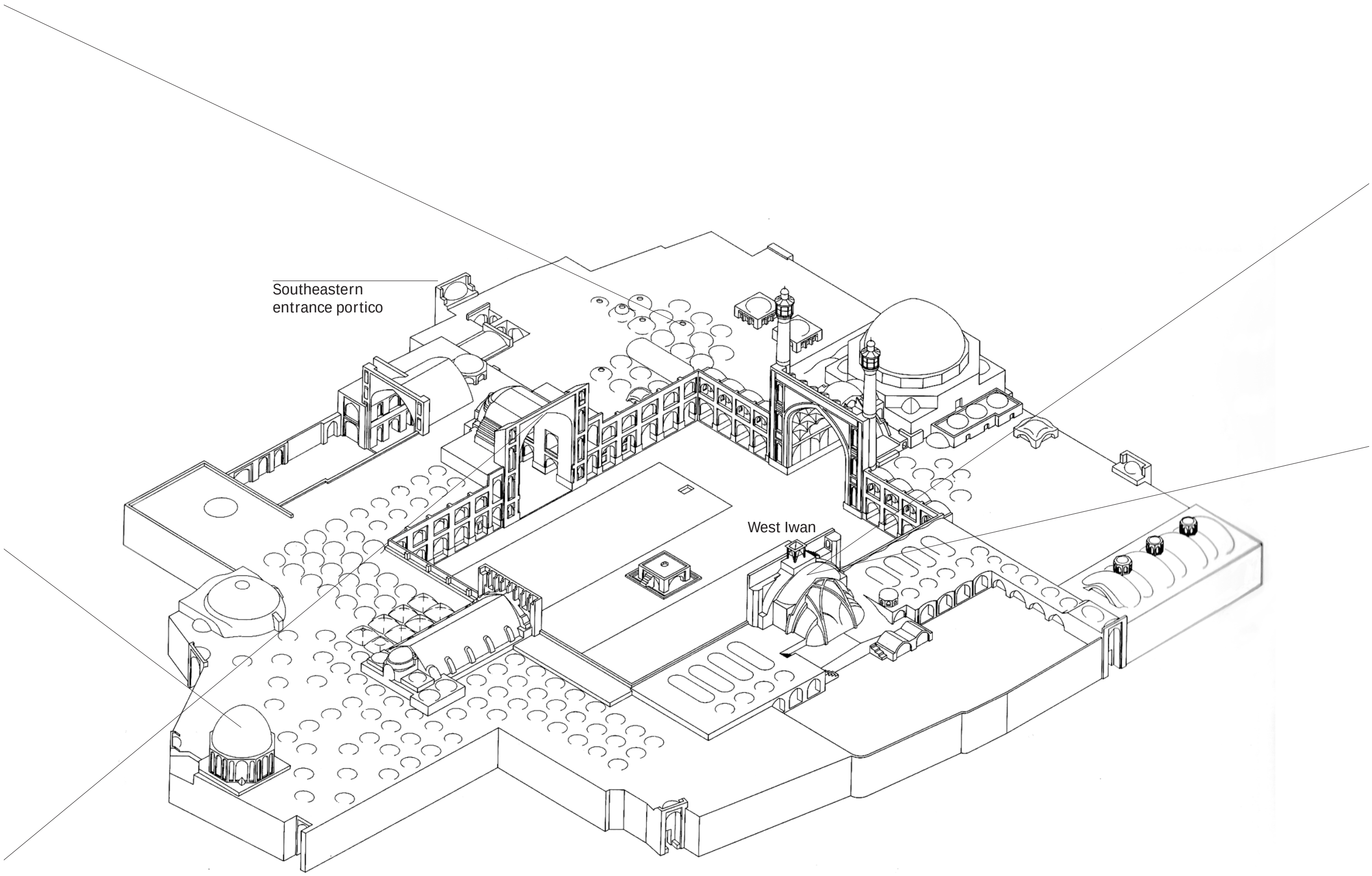
1 Axonometric of Friday Mosque from Ganjnameh: Cyclopedia of Iranian Islamic Architecture, Volume 7 (Congregational Mosques), Rowzaneh Publications, Tehran.

Hypostyle prayer hall - Buyid era (10th century)

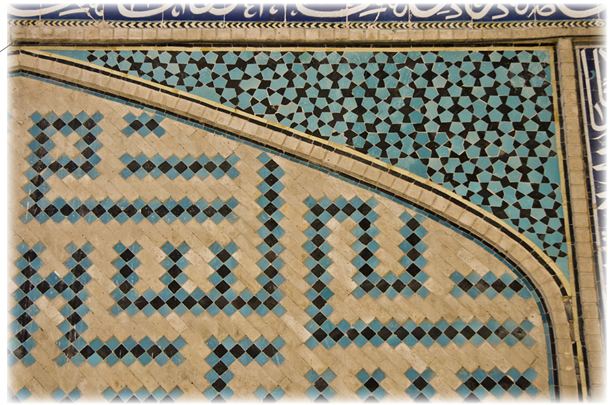
Taj-ol-Molk dome chamber -Seljuk era (11th century)

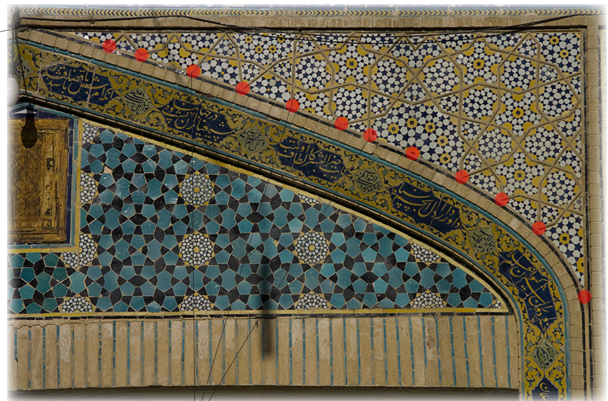


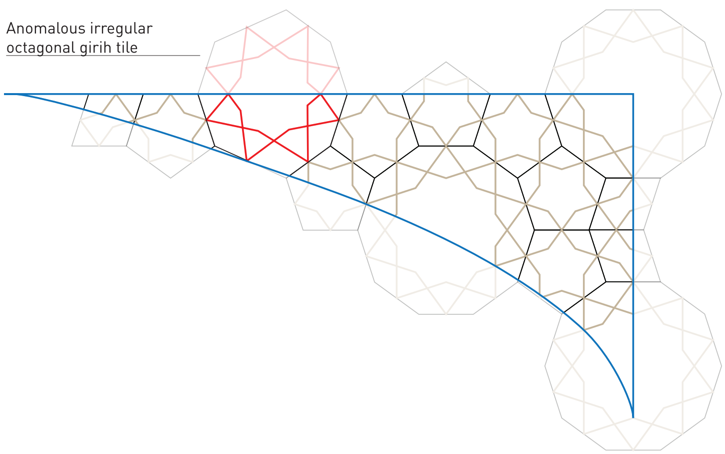
East Iwan muqarnas - Safavid era (16th century)


Southeastern
entrance portico
West Iwan

8




Nodal ‘Bézier control points’ linking the tiling pattern
to the arch geometry


Anomalous irregular
octagonal girih tile

Other decoration in the interior of the West Iwan also exhibits two spatial tiling scales, as illustrated in the girih tiling on page 8. These are similar to the Darb-e-Imam Shrine tilings discussed above. Again, as for the Darb-e-Imam tilings, there is a direct link between decoration and structure, where nodal points of the tiling act like Bézier control points for the arch geometry, such that the tiling geometry is driving the structural geometry.

However, the meta-tiling arrangement is clearly not periodic, and is more complex than the Darb-e-Imam tilings (which are periodic at the meta-tiling scale). The interior iwan meta-tiling in the Friday Mosque follows a quasiperiodic configuration. There is a significant discrepancy, however, where an octagonal meta-tile is inserted into the pentagonal quasiperiodic geometry. This is not an accident, but rather a deliberate breaking of the quasiperiodic symmetry. This compromise was made because a pure quasiperiodic meta-tiling does not fit seamlessly into the wedge-shaped frame at this scale. In fact, the pentagonal and octagonal geometries do not mesh consistently, and the octagonal meta-tile is an irregular (morphed) element.

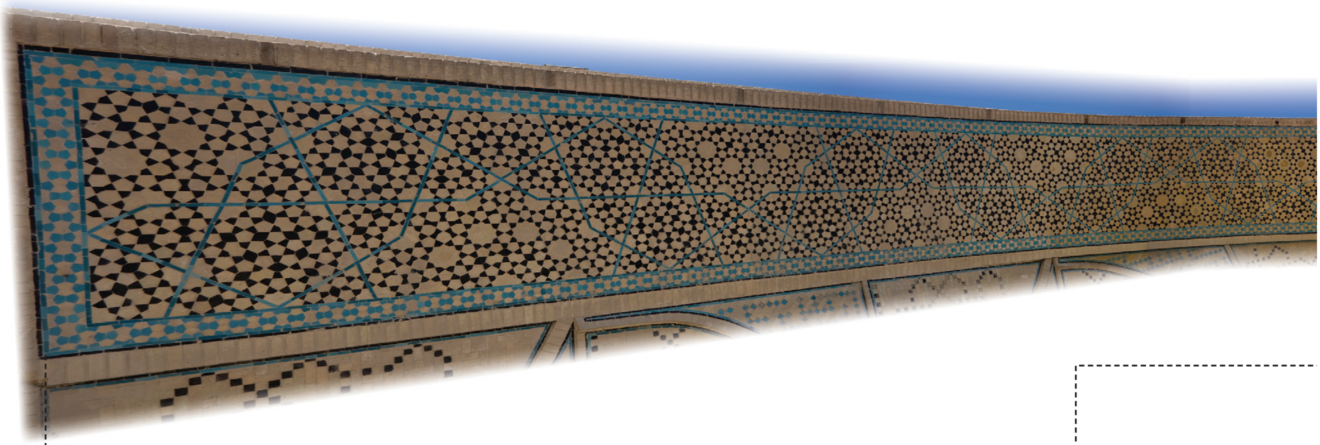
This signals a departure from the purity of the Darb-e-Imam tilings, which display a rigid adherence to the mathematical geometry. This discrepancy illustrates that the designers allowed an interplay between the design of the building structure and decoration, rather than a rigid adherence to the mathematical framework of the tiling system. While the tiling geometry often determined the structural geometry, in situations where inconsistencies arose, the structural geometry could force a reconfiguration of the tiling geometry.

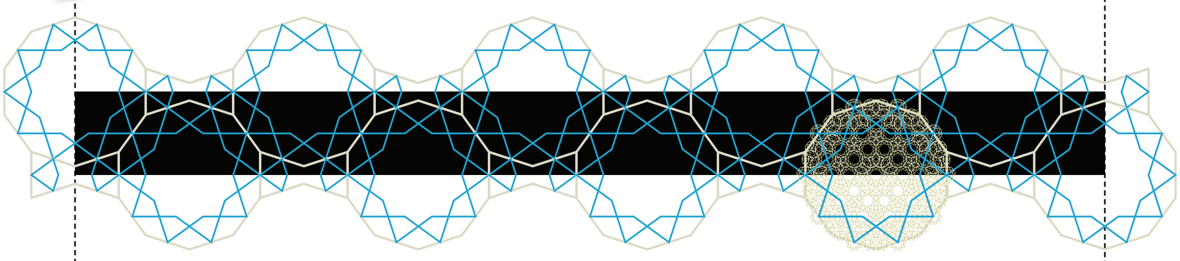
The fact that the same meta-tiling pattern is used in the tiling underneath the arch of the West Iwan of the Friday Mosque and in the Darb-e-Imam tilings suggests that they are contemporaneous. While an exact date for the West Iwan tilings of the Friday Mosque is not known, one possibility is that they were constructed at the same time as the the South Iwan was rebuilt by Uzun Hasan in 1475-76[2]. This conjecture is supported by similarities in style of several of the South and West Iwan tilings[3].

The southeastern entrance portico decoration marks a departure from both periodic and quasiperiodic tilings. This tiling can be constructed from girih tiles (as illustrated on page 13), but they are not arranged in either a periodic or quasiperiodic arrangement. This presents a more freeform approach to tiling, which is adapted to accommodate the structural form. In addition, the arch curve does not appear to be constrained by the tiling geometry as for the West Iwan and Darb-e-Imam tilings. This freeform approach is maintained in the colouring of individual tiles, where certain paths of connected pentagons are cyan-coloured. The choice of these paths is discretionary - they are constrained but not completely determined by the tiling geometry.

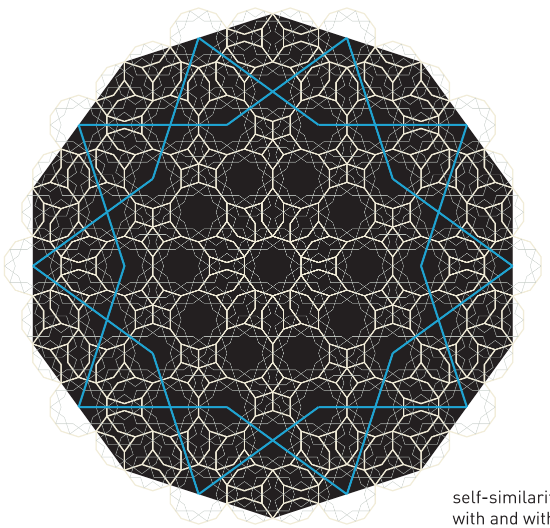
Again, an exact date for this tiling cannot be pinpointed, but an inscription[4] in the south-eastern entrance corridor (1530) places this as an early Safavid-era structure. If the tiling dates from the same era, then it marks a transition between the strict geometries of Timurid-era tilings and the more fluid forms of later Safavid masterpieces such as the dome of the Sheikh Lotfallah Mosque (1615).

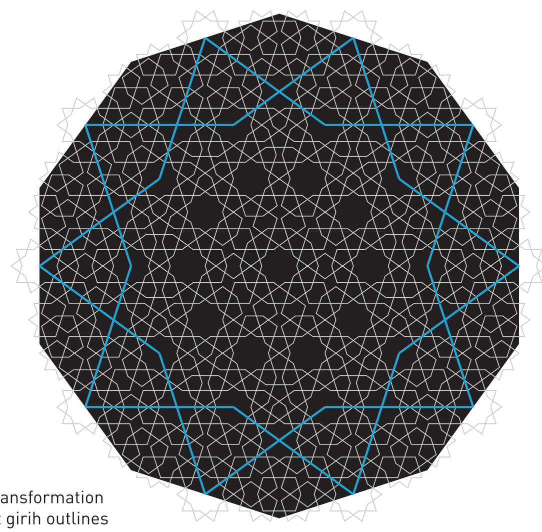
2 Golombek, L. & Wilber, D., 1988, The Timurid Architecture of Iran and Turan: Volume II Princeton: Princeton University Press, p.379.
3 In particular, the raised relief tilings.
4 Ganjnameh: Cyclopedia of Iranian Islamic Architecture, Volume 7 (Congregational Mosques), Rowzaneh Publications, Tehran, p. 118.





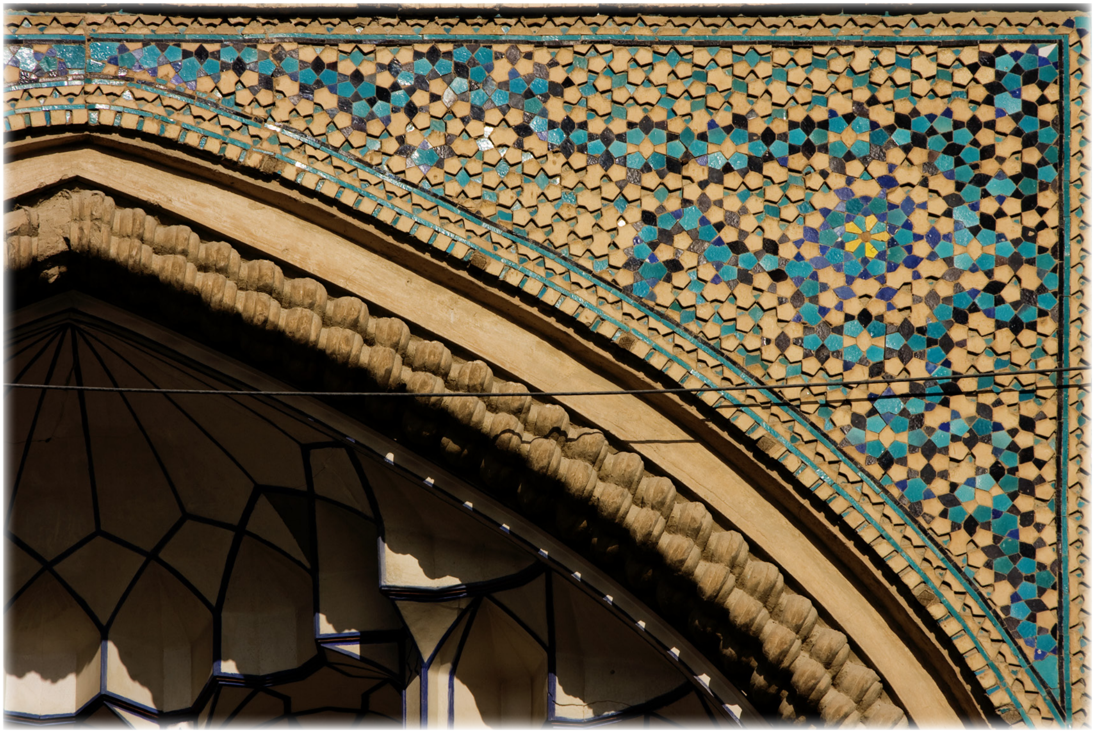
parametric relationship between structure (arch arc length) and decoration (meta-tiling scale)

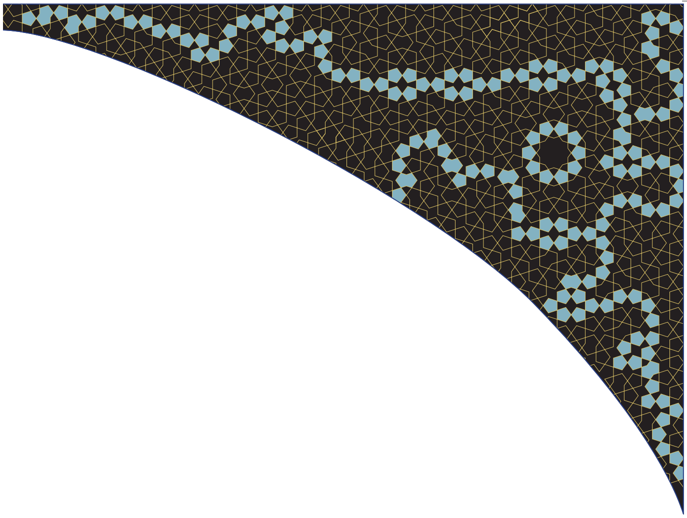




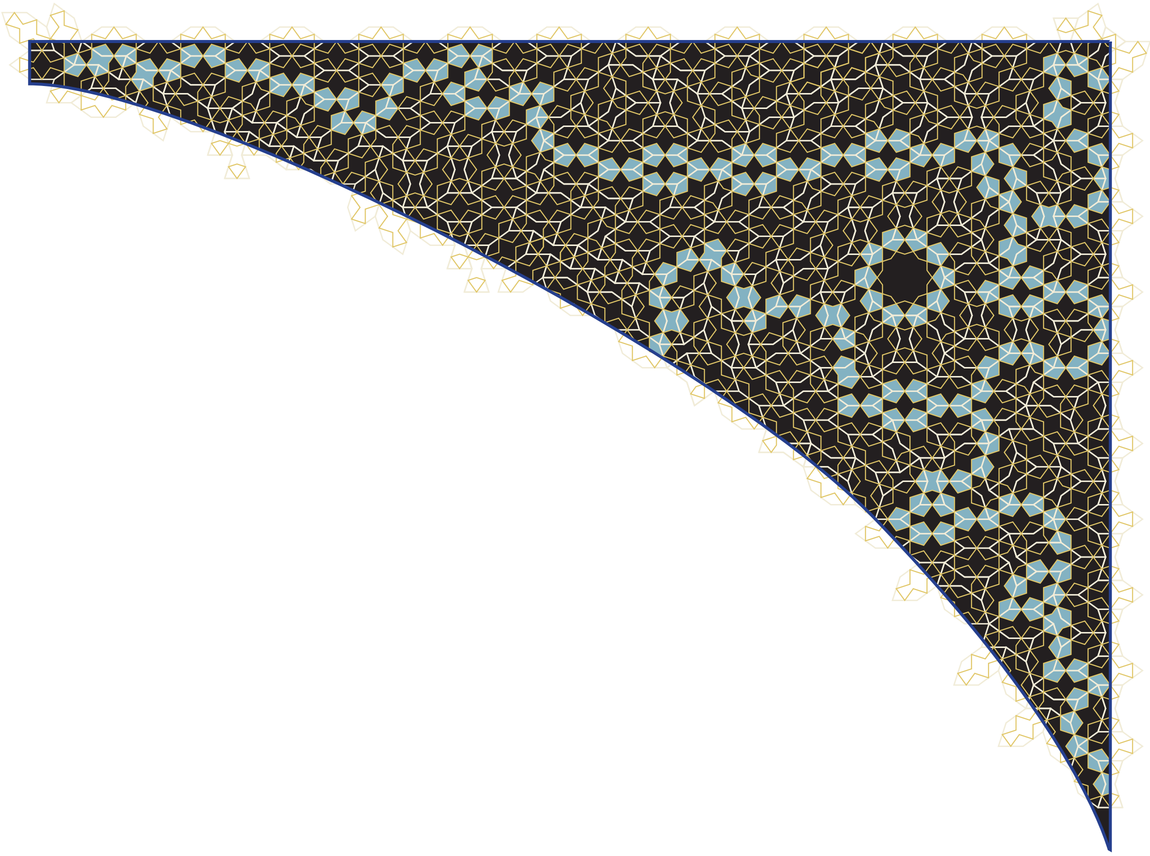
self-similarity transformation
with and without girih outlines

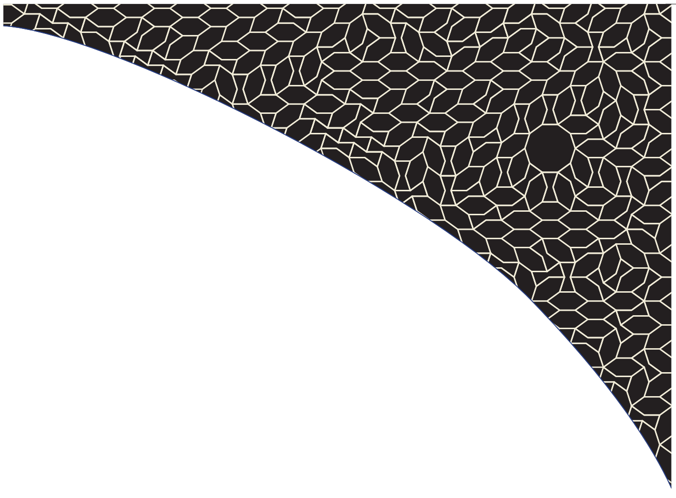
West Iwan, Friday Mosque, Esfahan.





Tiling pattern on the south-eastern entrance portico of the Friday Mosque, Esfahan.



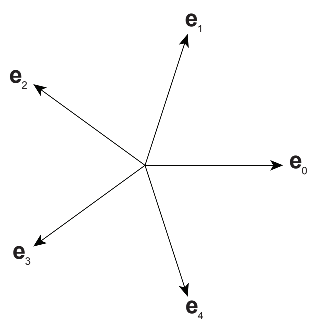


Configuration of girih tiles that generates the entrance portico tiling.

14 Two-dimensional Penrose tilings

A Penrose tiling can be generated by manually aligning Penrose tiles which obey specific matching rules. The downside of this method is that there is no guarantee that after positioning, say, one thousand tiles, that an impasse is reached and other permutations must be sought. There is also the possibility that the ‘error’ may have been made in positioning the first few tiles. Alternative algorithmic methods exist for generating Penrose tilings, including the deflation-rule, direct-projection, grid-projection, and generalized-dual methods[1]. All of these methods can be generalized to higher dimensional quasiperiodic tilings – the three-dimensional Penrose tiling will be discussed later in this report. The principal benefit of the generalized dual method, described below, is that it can generate every possible Penrose tiling together with more general quasiperiodic tilings, whereas the other methods can only generate a subset of all possible Penrose tilings.

The first step in generating a two-dimensional Penrose tiling via the generalized dual method is to set up a pentagrid - this can be thought of as the construction geometry. Each pentagrid consists of straight lines oriented normal to one of the five (pentagonal) star vectors:



$$\mathbf{e}_n = \{\cos(2\pi n/5)\ ,\ \sin(2\pi n/5)\}$$

for n = 0, 1, 2, 3, 4.

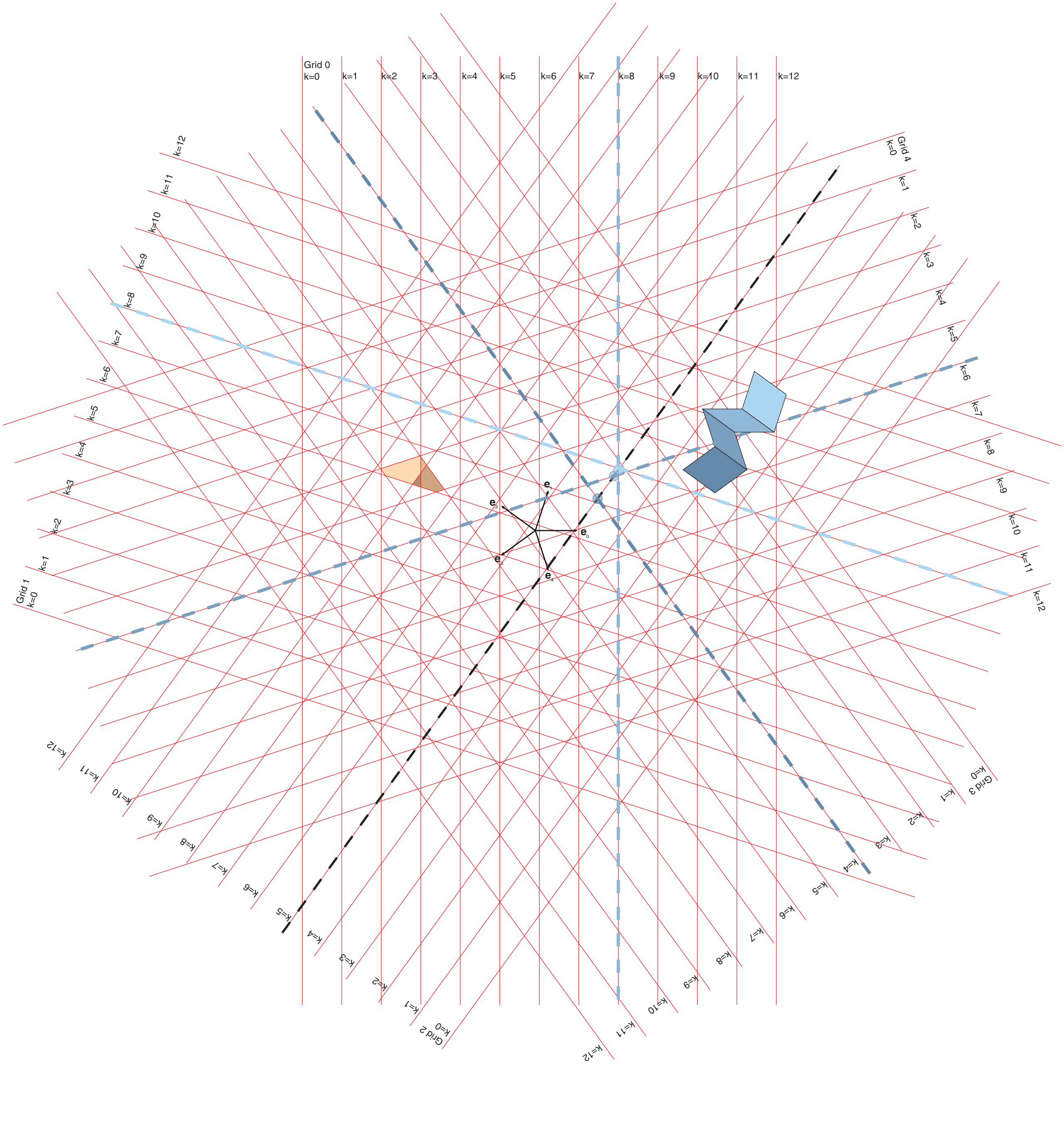
The grid shown to the right is a periodic pentagrid, where each grid spacing is constant. Each of the lines in each of the 5 grids (labeled by n=0,1,2,3,4) is labeled by the index number k=0,1,2, ... Each intersection point between grid lines has a one-to-one correspondence with a Penrose tile. The method to generate each Penrose tile is:

(1) Each grid intersection is surrounded by four open regions (for example, in this diagram, the four regions surrounding the intersection between grid 0: k=3 and grid 2: k=9 are coloured in shades of brown).

(2) Each open region maps to a vertex of the Penrose tile, and the four open regions then define the four vertices of the Penrose tile. The mapping is made by the mathematical operation of taking the dual. This is best explained by example:
In grid 0, the lightest brown open region lies between k=2 and k=3. Therefore k_0=2 (the lower value).
In grid 1, the open region lies between k=6 and k=7 and k_1=6.
In grid 2, the open region lies between k=9 and k=10 and k_2=9.
In grid 3, the open region lies between k=7 and k=8 and k_3=7.
In grid 4, the open region lies between k=4 and k=5 and k_4=4.
The dual then maps the open region to the tile vertex:

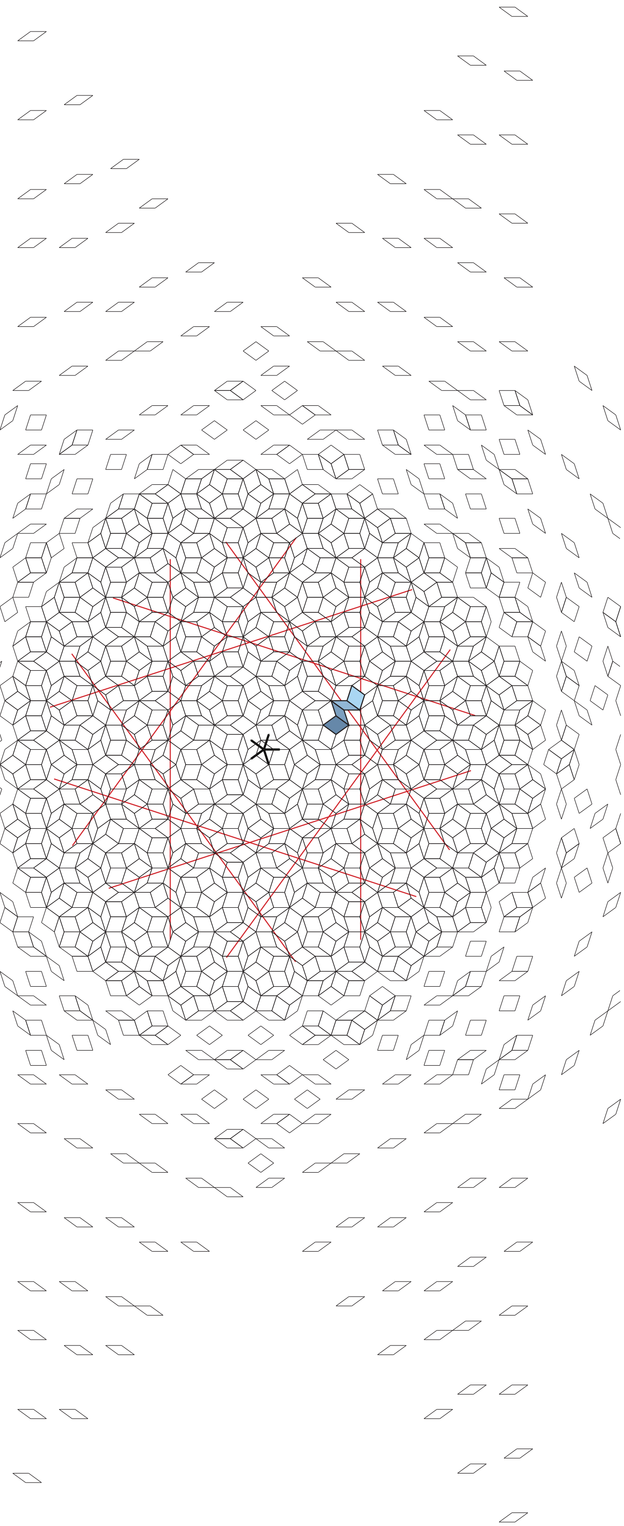
$$\mathbf{x} = k_0\,\mathbf{e}_0 + k_1\,\mathbf{e}_1 + k_2\,\mathbf{e}_2 + k_3\,\mathbf{e}_3 + k_4\,\mathbf{e}_4$$

1 Socolar & Steinhardt, 1986, Quasicrystals. II. Unit-cell configurations, Physical Review B, 34, 617-647.

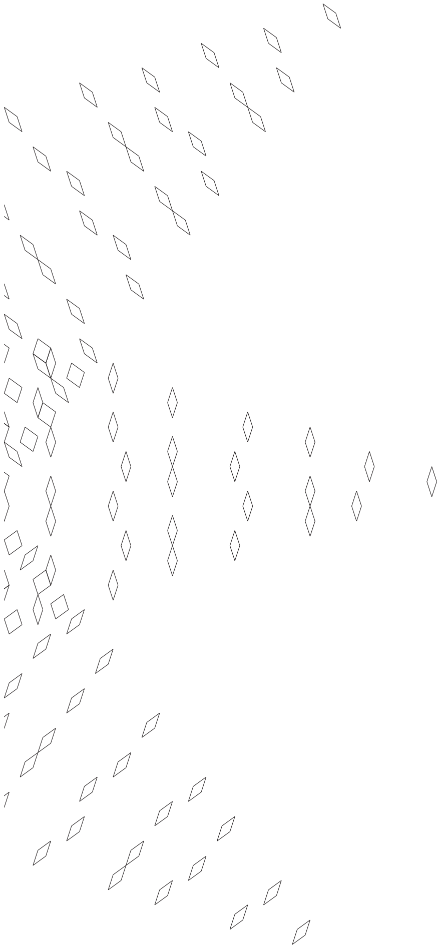


Grid 0
k=0
k=1
k=2
k=3
k=4
k=5
k=6
k=7
k=8
k=9
k=10
k=11
k=12
Grid 1
Grid 2
Grid 3
Grid 4
e_0
e_1
e_2
e_3
e_4

```
function (CoordinateSystem cs)
{
   for (int i = 0; i < 5; i++)
   {
      Point evec = new Point(this);
      evec.ByCartesianCoordinates(cs,Cos(72*i), Sin(72*i), 0 );
   }
}
Plane [][] function (Point origin, Point evec, int gridsize, double gridspacing, double gridphase)
{
Plane myplane = {};
Direction mydirection=new Direction();
for (int gridno = 0; gridno < 5; gridno++)
{
   myplane[gridno] ={};
   mydirection.ByOriginDirectionPoint(origin, evec[gridno]);
   for (int n = 0; n <= gridsize; n++)
   {
      myplane[gridno][n] = new Plane();
      myplane[gridno][n].ByDirectionAndDistanceFromOrigin(origin, mydirection, gridspacing*(n-gridsize/2+gridphase[gridno]));
   }
}
return myplane;
}
{baseCS,plane0,origin,starvector,gridsize,1.0,{gridphase0,gridphase1,gridphase2,gridphase3,gridphase4}}
Point [][][][][] function (CoordinateSystem cs, Plane starplane, Point myorigin, Point evec, int gridsize, double gridspacing, double gridphase)
{
Direction dplanes = new Direction();
Point intpoint = new Point();
Polygon mypoly = {};
Point tvec = {};
int kvec = {};
int index;
double tvecX= 0;
double tvecY= 0;
double tvecZ= 0;
   for (int j = 0; j < 5; j++)
   {
   tvec[j] = {};
      for (int N = 0; N <= gridsize; N++)
      {
      tvec[j][N] = {};
         for (int k = 0; k < 5; k++)
         {
         tvec[j][N][k] = {};
            if (k>j)
            {
               for (int M = 0; M <= gridsize; M++)
               {
               tvec[j][N][k][M] = {};
                dplanes.AtPlanePlaneIntersection(starplane[j][N], starplane[k][M] );
                intpoint.AtDirectionPlaneIntersection(dplanes, cs.XYPlane);
                  for (int l = 0; l < 5; l++)
                  {
                     if ((l!=j)&(l!=k))
                     {
                      kvec[l]=Floor((intpoint.X*evec[l].X+intpoint.Y*evec[l].Y+intpoint.Z*evec[l].Z)/gridspacing+gridsize/2 -gridphase[l]);
                     }
                  }
                  for (int jj = N; jj <= N+1; jj++)
                  {
                     for (int kk = M; kk <= M+1; kk++)
                     {
                      kvec[j] = jj-1;
                      kvec[k] = kk-1;
                      tvecX=0;
                      tvecY=0;
                      tvecZ=0;
                        for (int p = 0; p < 5; p++)
                        {
                         tvecX += kvec[p]*evec[p].X;
                         tvecY += kvec[p]*evec[p].Y;
                         tvecZ += kvec[p]*evec[p].Z;
                        }
                      index = 2*jj + kk;
                      tvec[j][N][k][M][index] = new Point();
                      tvec[j][N][k][M][index].ByCartesianCoordinates(cs, tvecX, tvecY, tvecZ );
                     }
                  }
               }
            }
         }
      }
   }
return tvec;
}
Polygon [][][][] function (CoordinateSystem cs, Point ppoint, int gridsize)
{
Polygon mypoly = {};
   for (int j = 0; j < 5; j++)
   {
   mypoly[j] = {};
      for (int N = 0; N <= gridsize; N++)
      {
      mypoly[j][N] = {};
         for (int k = 0; k < 5; k++)
         {
         mypoly[j][N][k] = {};
            if (k>j)
            {
               for (int M = 0; M <= gridsize; M++)
               {
                mypoly[j][N][k][M] = new Polygon();
                mypoly[j][N][k][M].ByVertices({ppoint[j][N][k][M][0],ppoint[j][N][k][M][2],ppoint[j][N][k][M][3],ppoint[j][N][k][M][1]});
               }
            }
         }
      }
   }
return mypoly;
}
```



(3) Repeat step 2 for all four open regions surrounding the intersection point to give the four vertices of the corresponding Penrose tile.

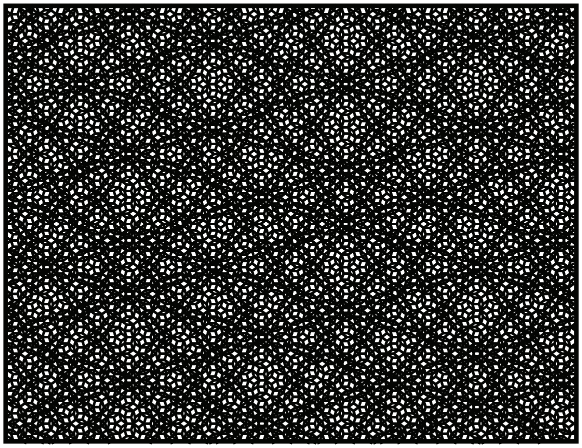


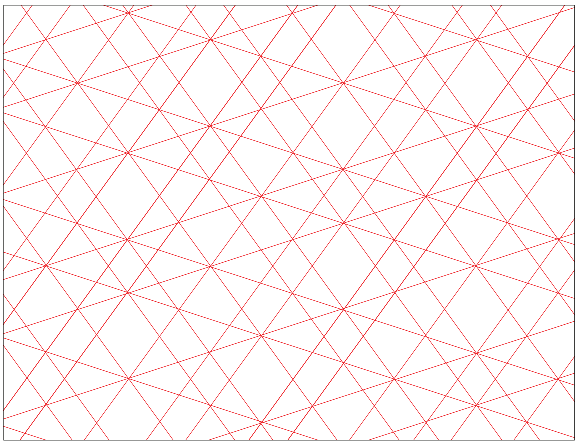
The blue Penrose tiles in this diagram show the four Penrose tiles generated by the dual method for four adjacent intersection points along the grid 2: k=5 line (dashed black line). In order, the intersecting lines are grid 3: k=4 (dark blue dashed line), grid 4: k=6, grid 0: k=8, and grid 1: k=8 (light blue). The four generated Penrose tiles are also adjacent because the common vertices correspond to the common open region adjoining neighbouring intersection points.

This algorithmic process can be automated - Generative Components was utilized here for this purpose. This diagram shows a Penrose tiling for grid index values $k_n \leq 10$. The extent of the pentagrid (red lines) is also displayed. As more grid lines are added to the pentagrid (with $k_n > 10$) the tiling expands. In the limit as k_n approaches infinity, the Penrose tiling covers the entire plane, with infinite variation and with no gaps or overlaps.

Alternative Penrose tilings are generated by shifting the phase of each (periodic) grid. As each grid phase is varied the grid intersection points shift, and the duals map to new Penrose tile vertex positions. The constraint that distinguishes the subset of Penrose tilings from the larger set of quasiperiodic tilings is that the five grid phases must add up to zero.

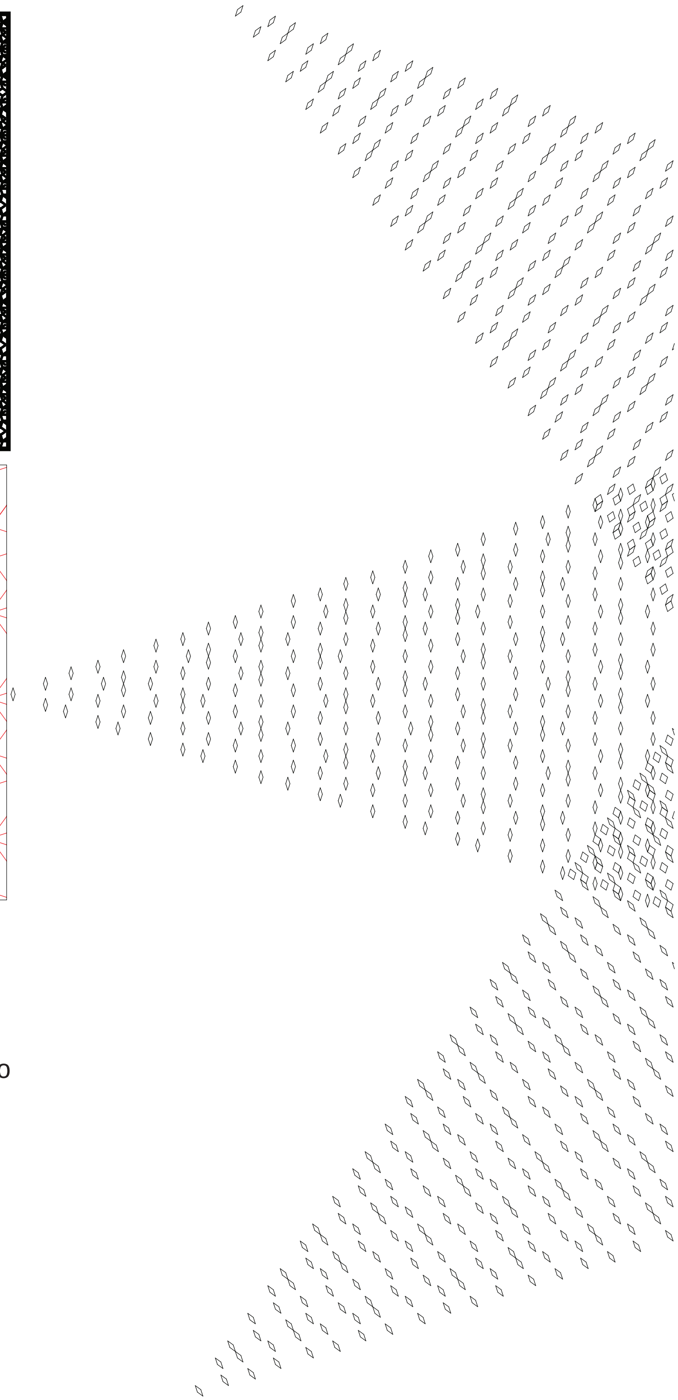
The quasiperiodic nature of the Penrose tiling can be revealed by making a copy of a particular Penrose tiling and translating it relative to of the original tiling (imagine that the tilings are drawn on acetate sheets and one sheet is moved across the other). The resulting Moiré pattern, where the two patterns interfere constructively and destructively, consists of bars aligned normal to the star vectors (and parallel to the pentagrid lines), but where the grid spacing consists of a sequence of long and short spacings. Each of

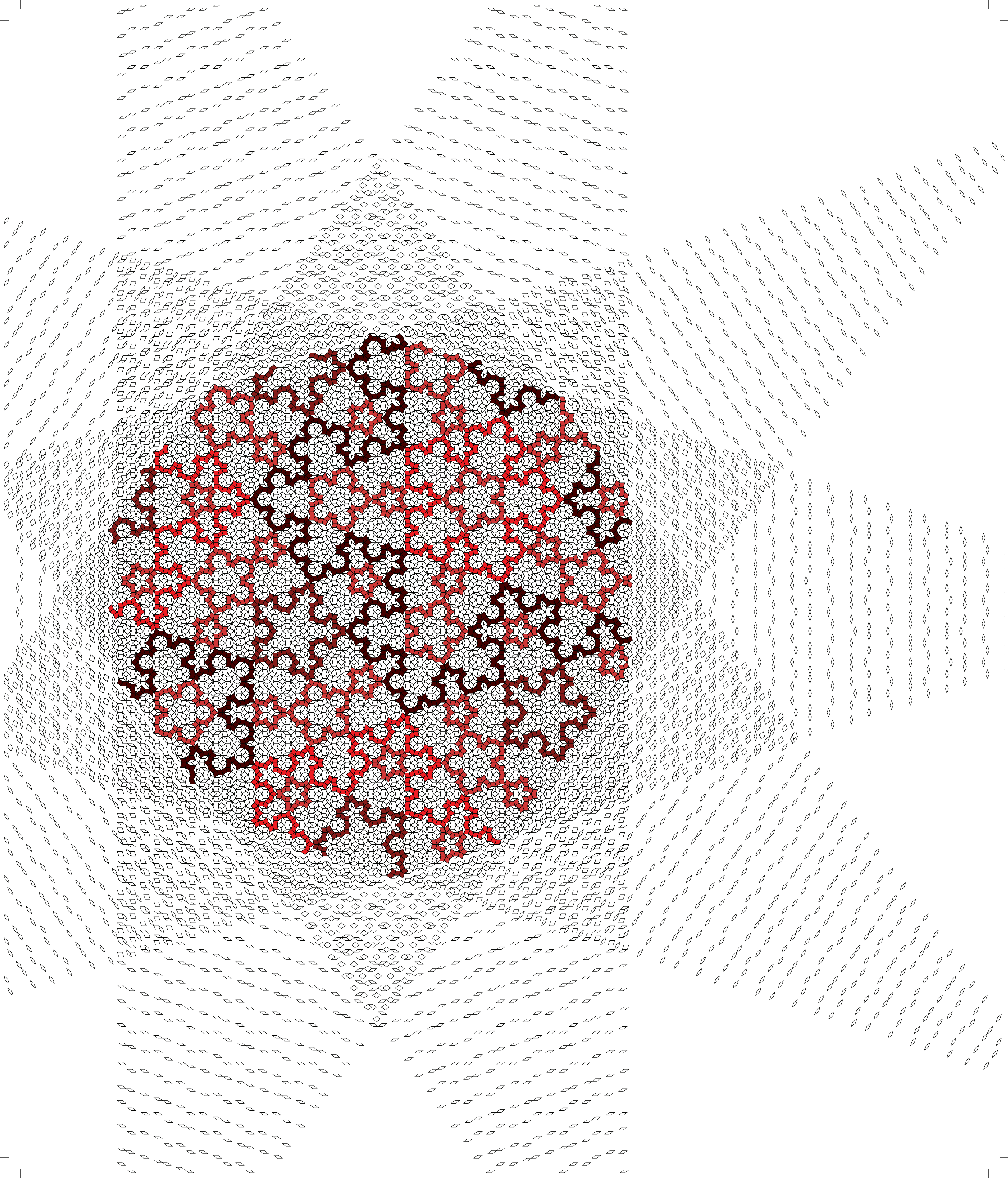


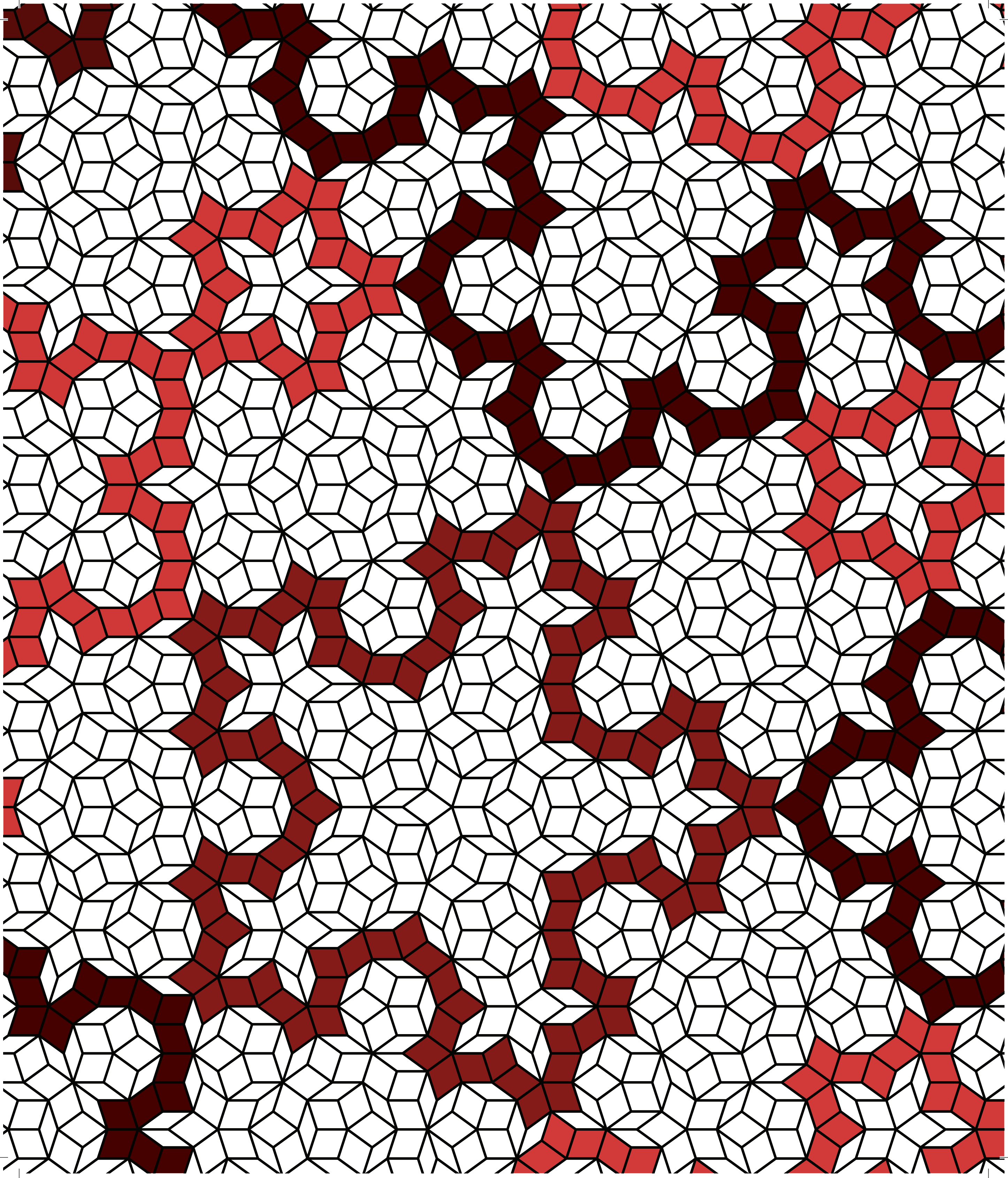


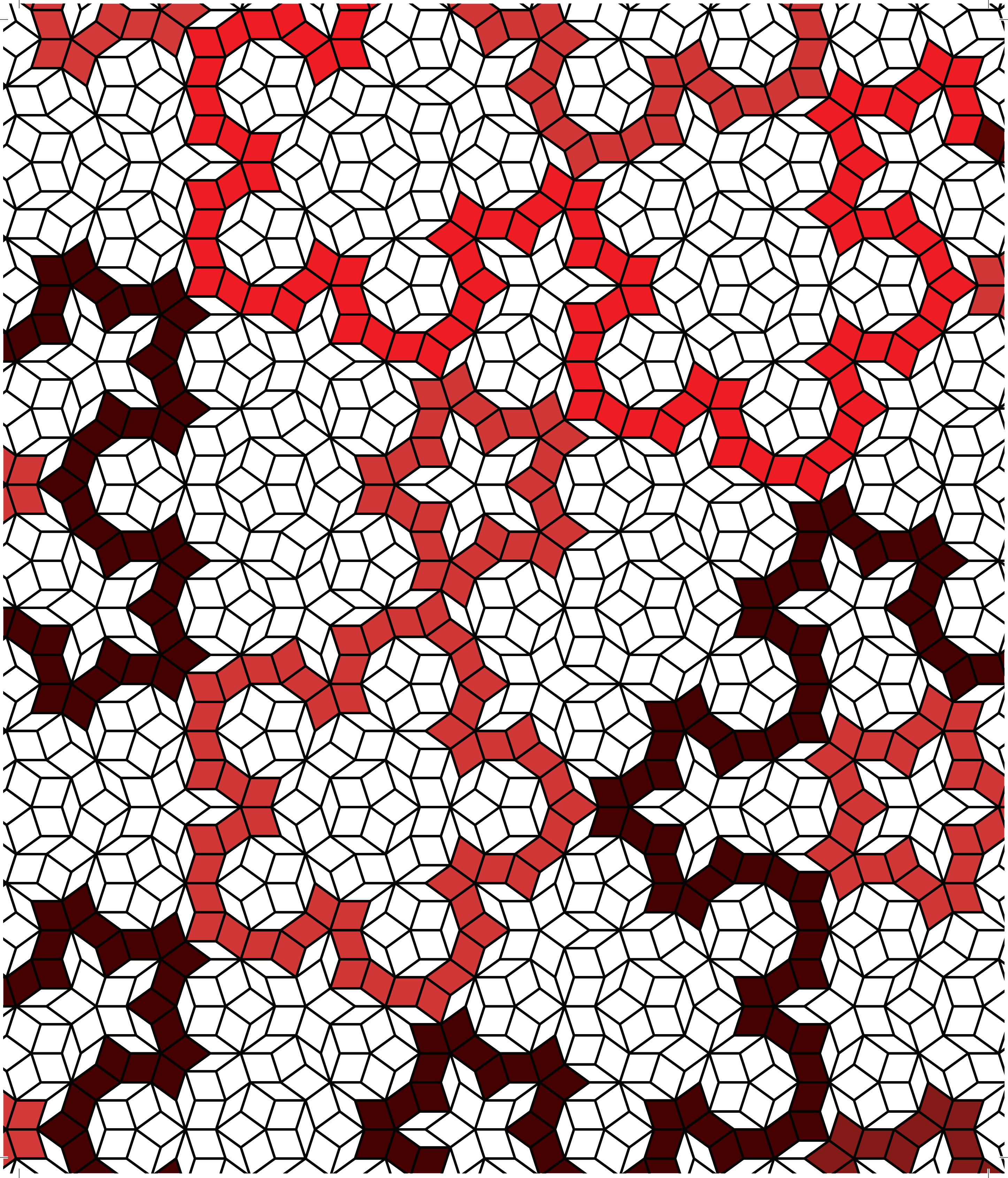
these binary sequences is quasiperiodic. The lighter regions in the Moiré pattern correspond to where the overlapping tilings are locally similar.

The rich complexity of the Penrose tiling can also be revealed by following the lead of the tilers of the southeastern entrance portico of the Friday Mosque in Esfahan, and tracing the connected path of one tile shape. The patterns overlaid on the Penrose tiling on page 19 are made by tracing the paths of connected 'fat' Penrose tiles. One of the properties which differentiate Penrose tilings from other quasiperiodic tilings is that these paths do not diverge - there is only one route to follow.

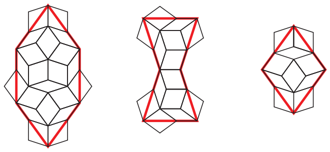


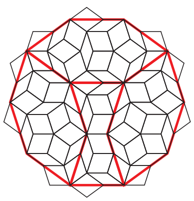




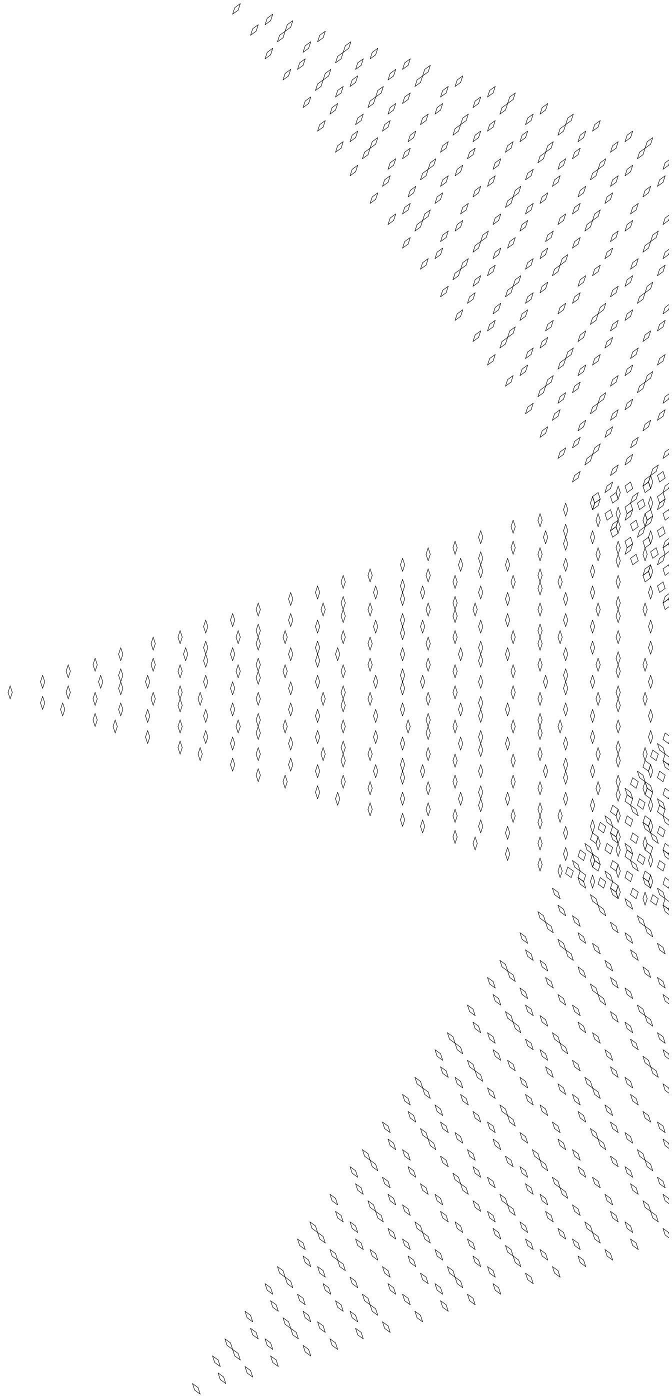


There is a direct relationship between Penrose tiles and the Persian girih tiles, as illustrated below. Several of the Penrose tiles are bisected at the girih-tile boundaries and naturally provide matching rules for connecting girih tiles[1]. The decagon girih tile is redundant in the Penrose conversion, and can be constructed from a combination of two other girih tiles, as shown.

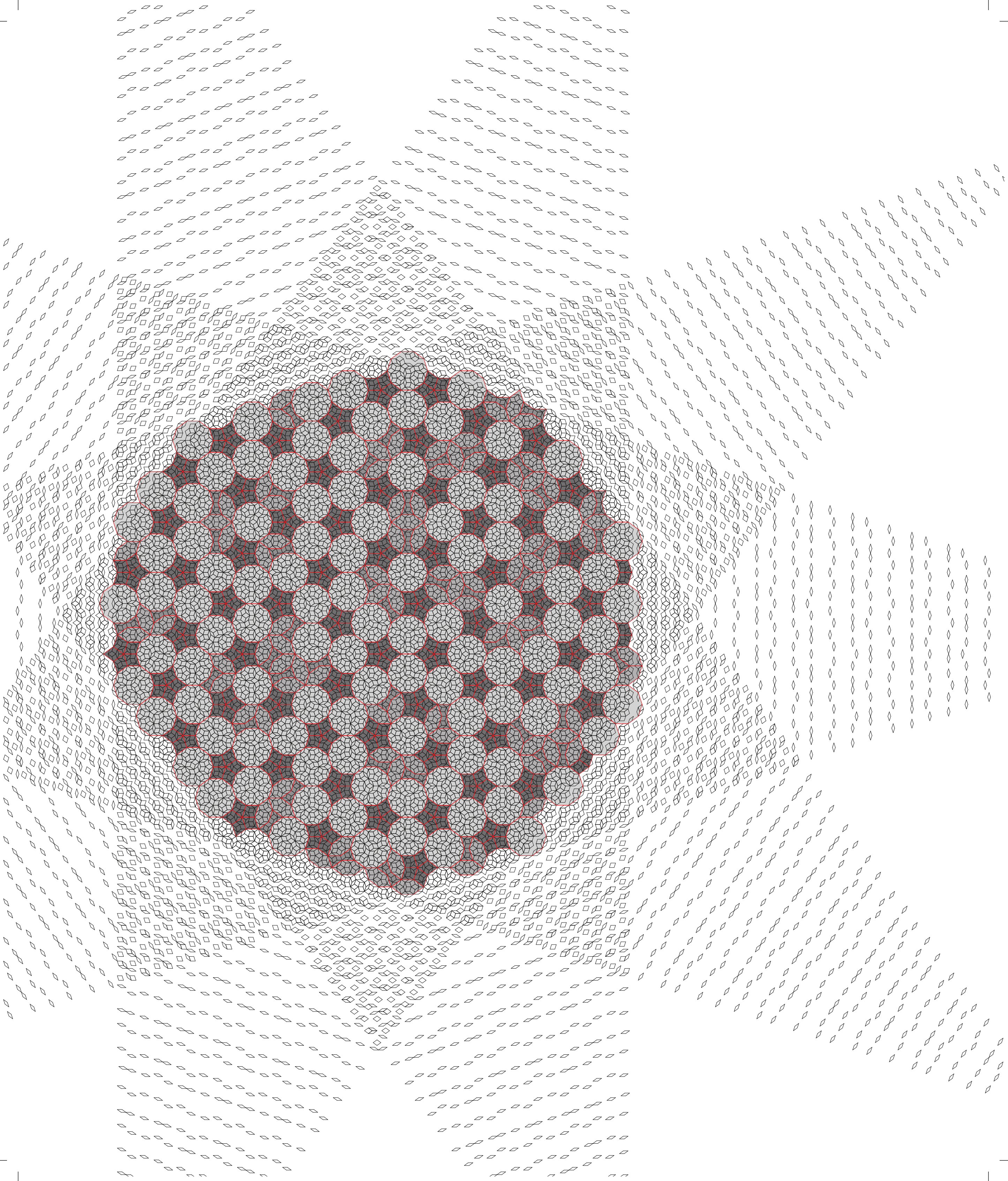




The girih tiling pattern corresponding to a Penrose tiling is shown here. This demonstrates a contemporary technique to generate large-scale quasiperiodic girih tilings. The strapwork pattern corresponding to a portion of this girih tiling is shown on page 5.

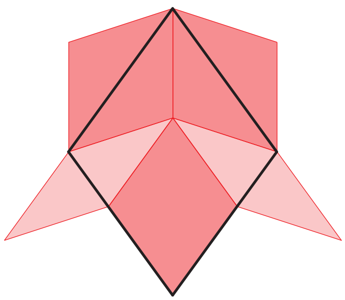


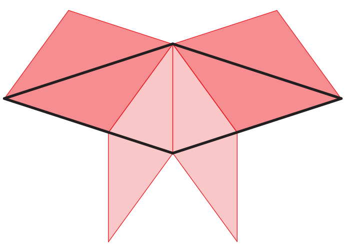
1 Note that in the alternative ‘kites and darts’ formulation, the Penrose tile shapes fit exactly inside the girih-tile boundaries (see Lu & Steinhardt, 2007).



24 Self-similarity transformations

Another remarkable property of Penrose (and girih) tiles is their intrinsic self-similarity. Penrose tiles can be ‘deflated’ to smaller scales (or inflated to larger scales) via a recursive substitution operation. In the deflation operation, each Penrose tile is broken down into a cluster of Penrose tiles at a smaller scale, such that the entire Penrose tiling is deflated into a different Penrose tiling at the smaller scale. The deflation operation does not respect the boundaries of individual Penrose tiles, but it does respect half-tiles. This is also evident in the Persian quasiperiodic tilings discussed above, where half-tiles often line the outside boundary of the tiling.





Deflated Penrose tilings can also be generated by an algorithmic process using the generalized dual method outlined above. However this cannot be achieved with periodic pentagrids for the construction geometry, as outlined in the previous section. Instead, quasiperiodic pentagrids must be used. One specific quasiperiodic pentagrid, the Fibonacci pentagrid, is used to generate Penrose tilings[1], with star vectors:

$$\mathbf{e}_n = \{\cos(2\pi n/5)\ ,\ \sin(2\pi n/5)\}$$

for grid number n = 0, 1, 2, 3, 4, and line positions given by

$$x_{nN} = N + \alpha_n + (1/\tau) * \mathrm{Floor}[N/\tau + \beta_n]$$

where N is the grid index number, τ is the golden ratio, the floor function returns the closest integer less than the rational argument, and

$$\alpha_n = (6\tau-1)/(2\tau+4), \quad \beta_n = -2/(\tau+2), \quad \text{for all n.}$$

This generates one specific Penrose tiling (the special case with a centre of symmetry). All other Penrose tilings can be generated by making a specific transformation of the α_n and β_ns[2].

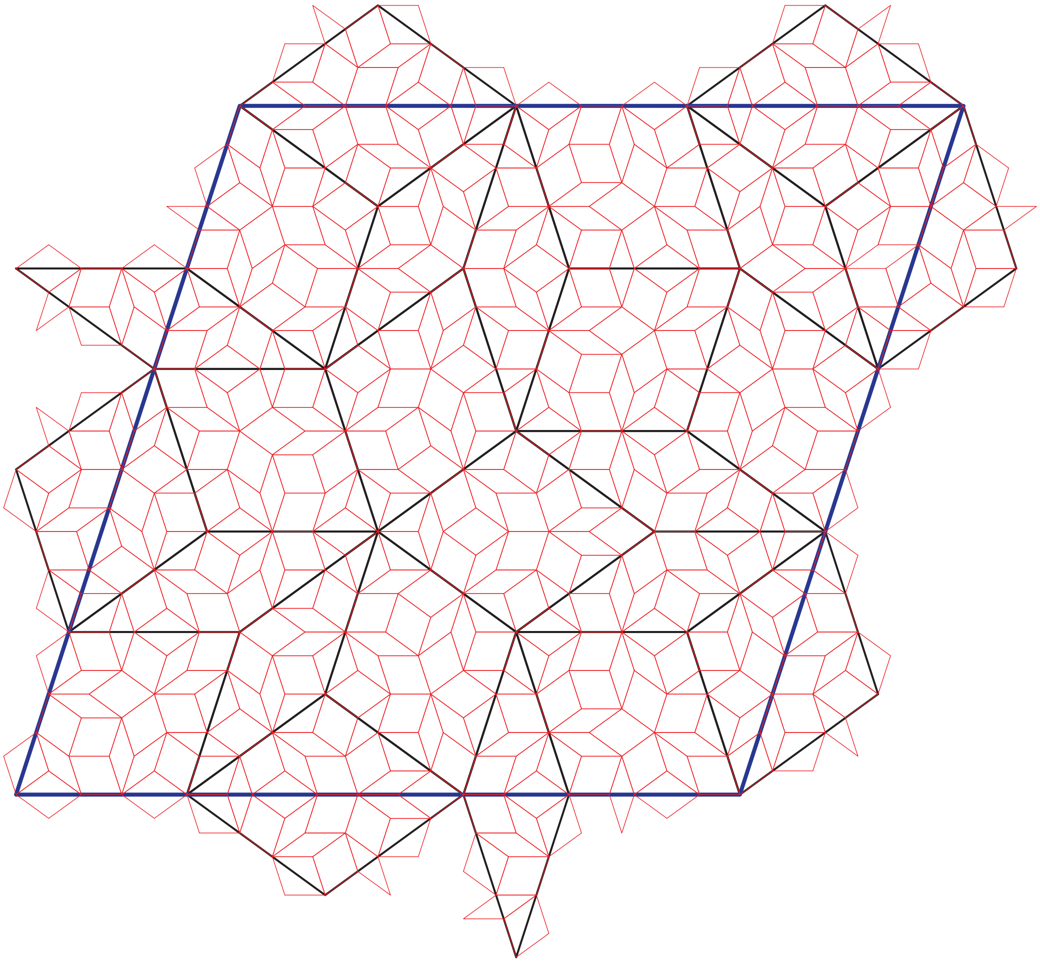
The deflated Penrose tiling is generated by making the transformation to α^*_n and β^*_n (where * denotes the deflated tiling), with

$$\alpha^*_n = \tau\ \{\alpha + \mathrm{Floor}[\beta]/\tau - 1/2\} - \mathrm{Floor}[\beta^*]/\tau$$

$$\beta^*_n = \{\mathrm{Floor}[\beta] - \beta\}/\tau$$

The deflation process can be repeated recursively to arbitrarily small scales. The diagram to the right shows a triple deflation from the blue Penrose tile to the black tiles, and a further triple deflation from the black to the red tiles. The shift in scale from the meta-tiling to the individual tile scale in the Timurid-period tilings discussed above corresponds to a triple deflation.

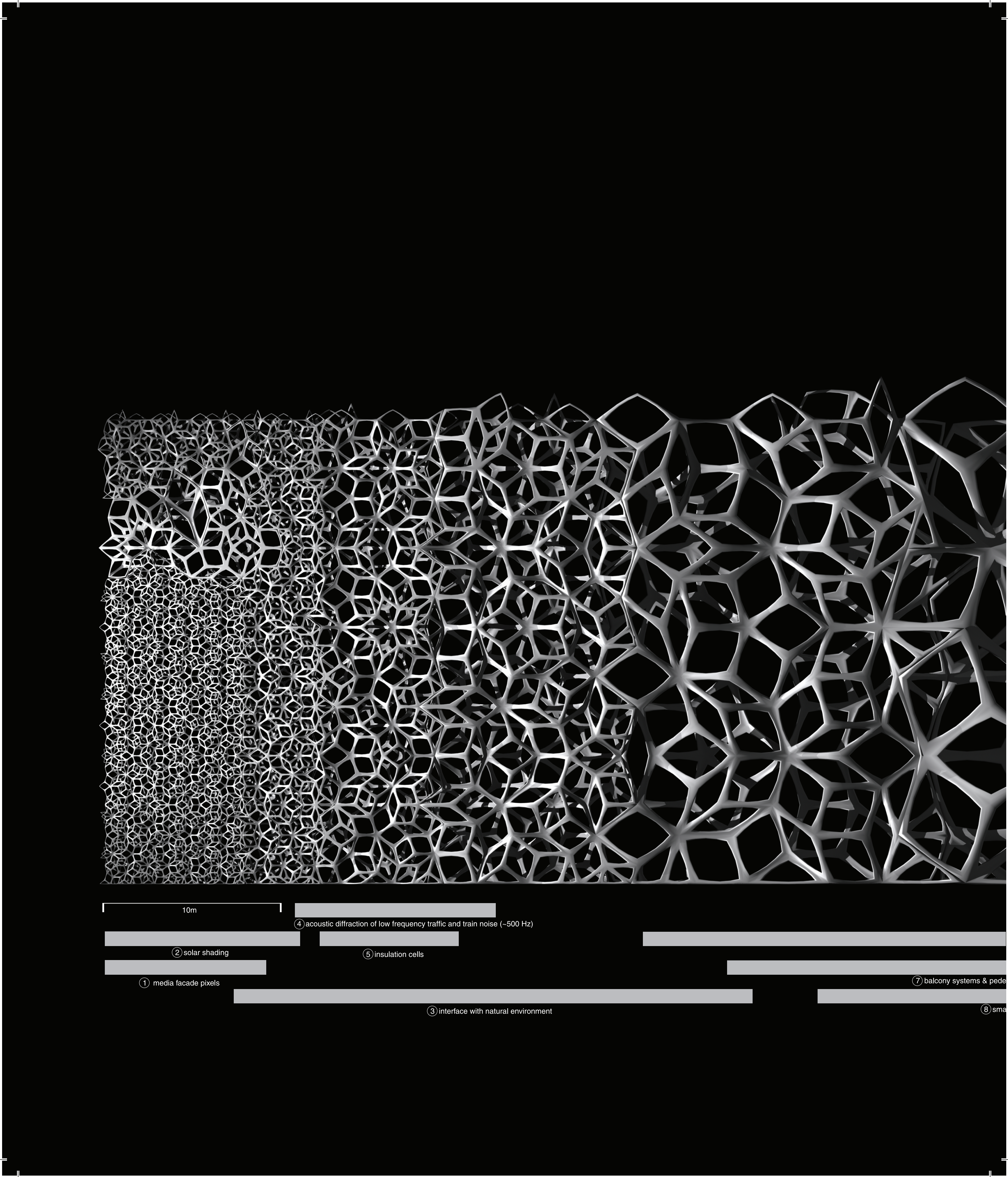
1 Socolar & Steinhardt, 1986, Quasicrystals. II. Unit-cell configurations, Physical Review B, 34, 617-.647.
2 See Socolar & Steinhardt (1986), eq. 19.



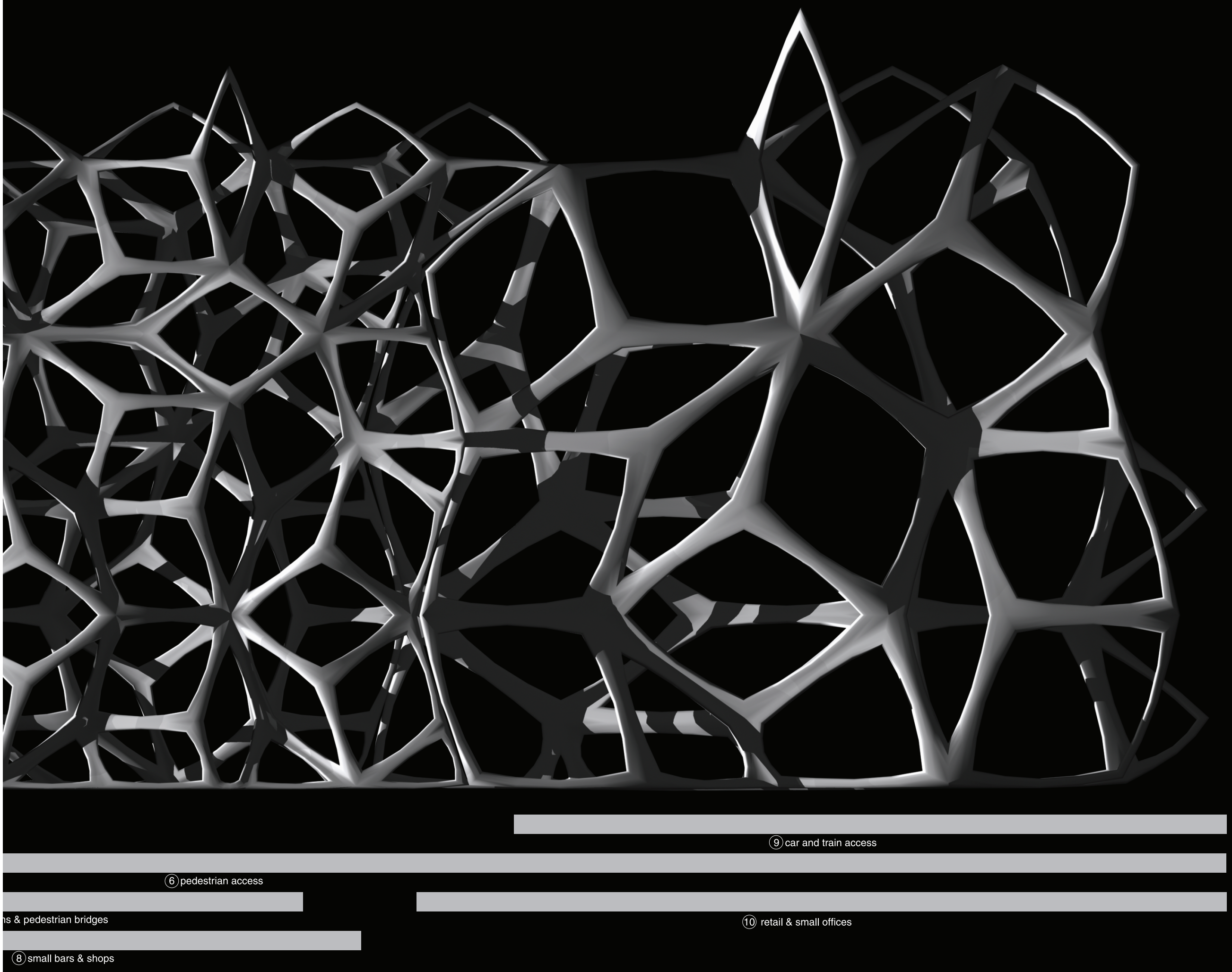
The following pages illustrate a contemporary an urban-scale mashrabiah which exploits the self-similarity properties of Penrose tiles on a functional basis.

On pages 26-27, aperformance-to-scale elevation denotes how various scales of the screen elements offer variable environmental responses.

The images on pages 28-29 shows this structure embedded and dispersed in an urban streetscape.



10m
4 acoustic diffraction of low frequency traffic and train noise (~500 Hz)
2 solar shading
5 insulation cells
1 media facade pixels
7 balcony systems & pede
3 interface with natural environment
8 sma


9 car and train access
6 pedestrian access
ns & pedestrian bridges
10 retail & small offices
8 small bars & shops

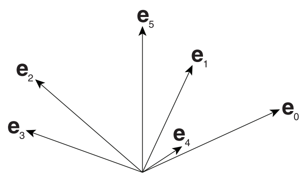

LEARN
MAKE
FITNESSCENTER
SPINNING
ROSENTHALER

Three-dimensional quasiperiodic tilings

Three dimensional quasiperiodic tilings can also be generated via the generalized dual method. For three-dimensional Penrose tilings the star vector is icosahedral, with:

$$\mathbf{e}_n = \{2\cos(2\pi n/5),\ 2\sin(2\pi n/5),\ 1\}\ /\sqrt{5}$$

for $n = 0, 1, 2, 3, 4$, and $\mathbf{e}_5 = (0\ ,\ 0\ ,\ 1)$.



Three-dimensional quasiperiodic tiles generated from this icosahedral star vector are oblate and prolate rhombohedra, analogous to the fat and thin two-dimensional Penrose tiles. It is necessary to generate a three-dimensional Penrose tiling from a quasiperiodic hexagrid[1], whereas a two-dimensional Penrose tiling can be generated from periodic or quasiperiodic pentagrids. Periodic hexagrids will generate three-dimensional quasiperiodic tilings, but not a three-dimensional Penrose tiling, which is a specific subclass of all possible three-dimensional quasiperiodic tilings. The grid lines in two dimensions generalize to grid planes in three dimensions (as shown in the figure below) and the point of intersection between two lines in the two-dimensional case becomes the point of intersection between three planes. The quasiperiodic spacing between planes (index number N) parallel to the star vector $\mathbf{e}_n$ satisfies the formula:

$$\mathbf{x}\cdot\mathbf{e}_n = x_{nN} = N + \alpha_n + (1/\tau) * \mathrm{Floor}[N/\tau + \beta_n]\ .$$

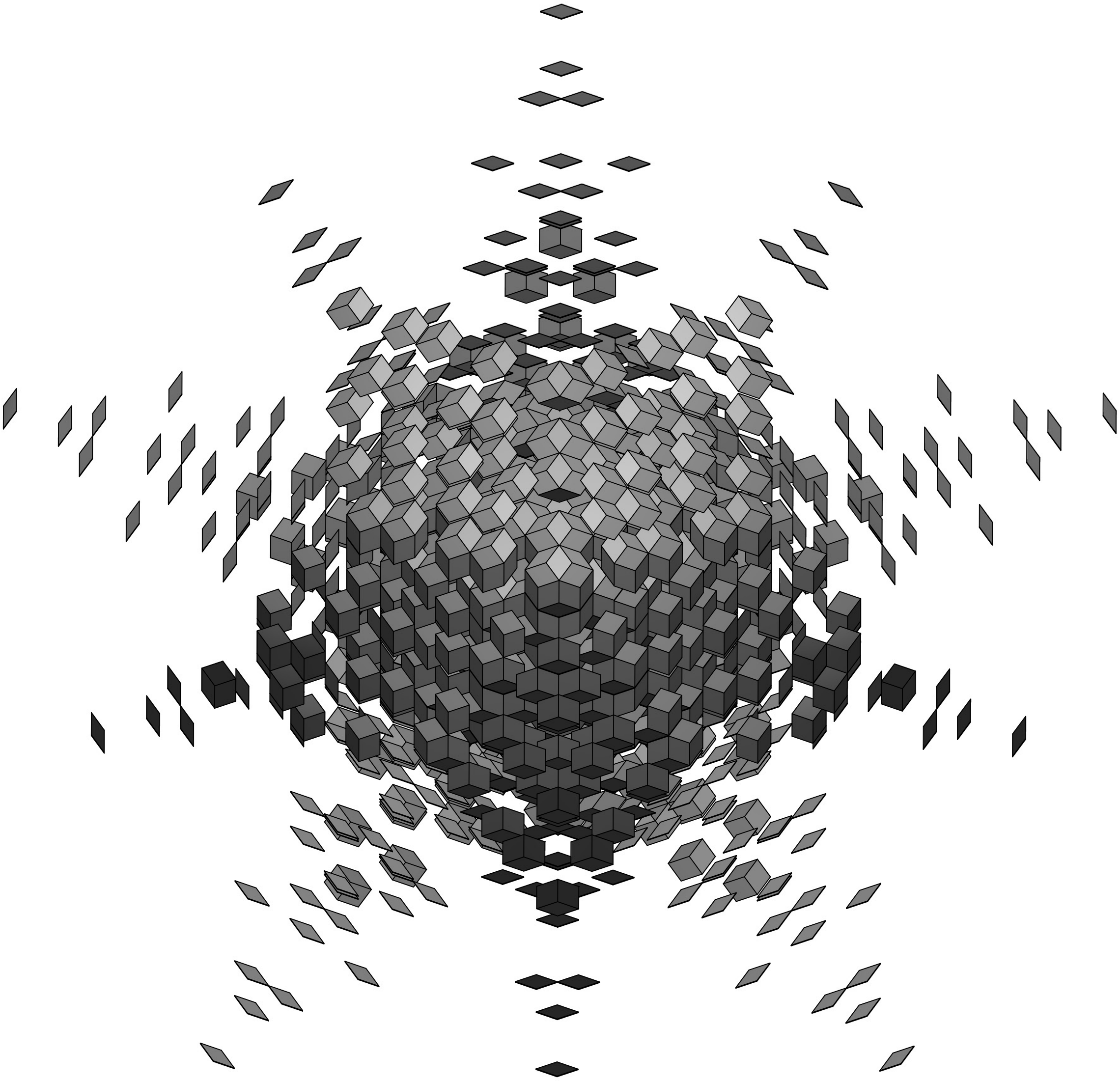
For the special case of a Penrose tiling, $\alpha_n = 1/\tau$ and $\beta_n = -1/2$ for all n.

Three dimensional quasiperiodic tilings can also be deflated, using the same formulas given on page 24. A quasiperiodic tiling spanning three scales is illustrated overleaf.

In a three-dimensional Penrose tiling, the tiles are always arranged in four clusters or unit cells. This is related to the highly singular nature of the quasiperiodic hexagrid, where more than three planes often meet at each intersection point. The smallest unit cell, the prolate rhomohedron, corresponds to the non-singular intersections at points where three planes meet. The other clusters, the dodecahedron, the icosahedron, and the tricontahedron, correspond to four-fold, five-fold, and six-fold (where all six planes intersect at one point) singularities, respectively[2]. These four zonohedra are the constituent elements of a three-dimensional Penrose tiling. Each zonohedron can be decomposed into prolate and oblate rhombohedra by slightly

1 Socolar & Steinhardt (1986).

2 Socolar & Steinhardt (1986).

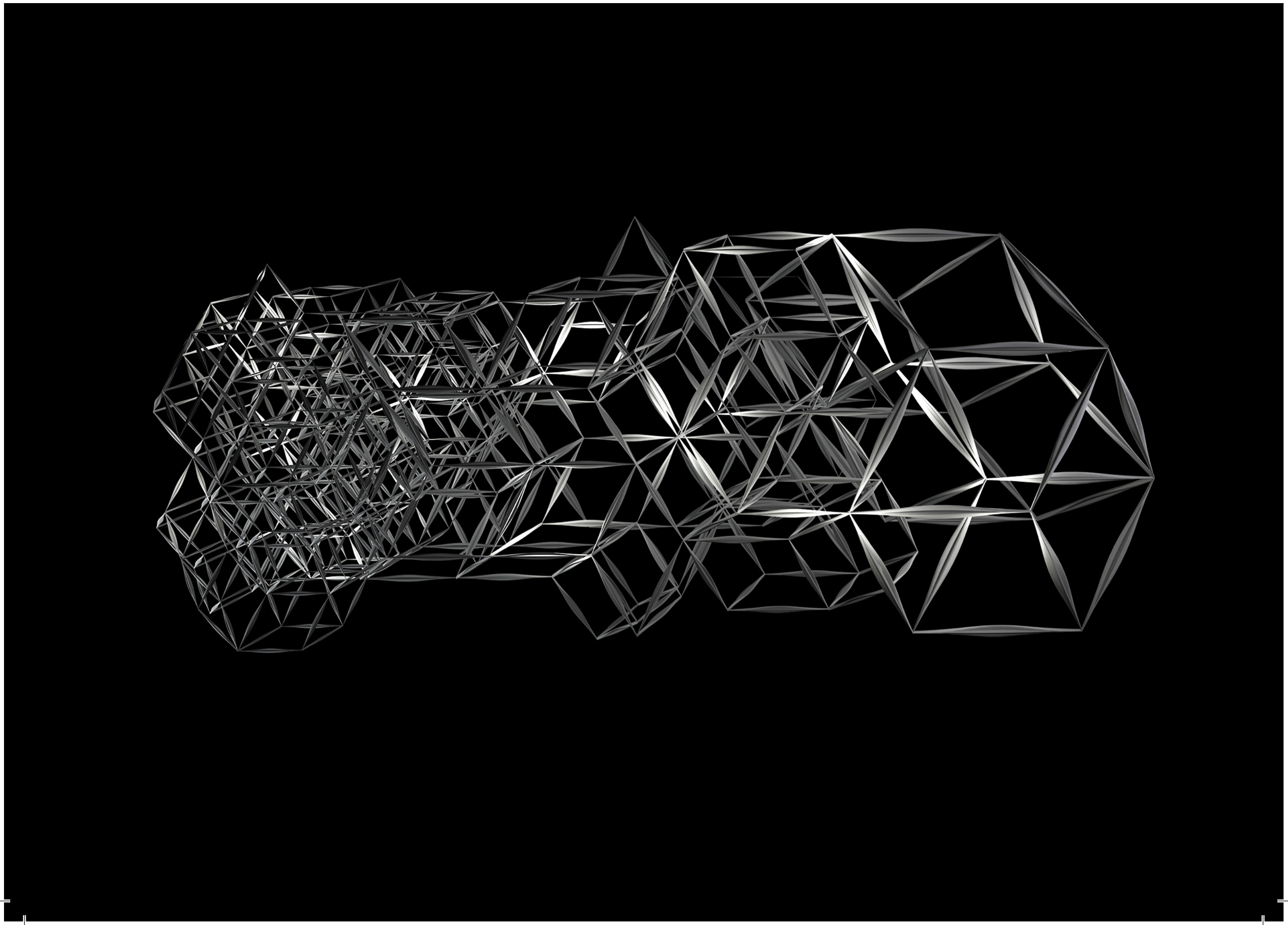


shifting each plane grid in the hexagrid, but by doing so the symmetry is broken.

The Penrose unit cells must obey certain matching rules when assembled to form a three-dimensional Penrose tiling. Some assemblages obeying the matching rules are shown on the following page.

Three-dimensional Penrose tiles

Quasiperiodic tiling with two deflations



References

Ganjnameh: Cyclopedia of Iranian Islamic Architecture, Volume 7 (Congregational Mosques), Rowzaneh Publications, Tehran.

Golombek, L. & Wilber, D., 1988, The Timurid Architecture of Iran and Turan: Volume II Princeton: Princeton University Press.

Lu, P.J. & Steinhardt, P.J., 2007, Decagonal and Quasi-Crystalline Tilings in Medieval Islamic Architecture, Science, 315, 1106-1110.

Makovicky, E., 1992, in Fivefold Symmetry, Ed. I. Hargittai, Singapore: World Scientific, pp.67-86.

Penrose, R., 1989, The Emperor's New Mind, Oxford: Oxford University Press.

Socolar, J.E.S. & Steinhardt, P.J., 1986, Quasicrystals. II. Unit-cell configurations, Physical Review B, 34, 617-647.

Acknowledgements

My research trip to Iran in January-February 2008 was funded by the NSW Architects Registration Board (2007 Byera Hadley Travelling Scholarship for Students).

I am grateful to the following people in Australia and Iran for valuable travel advice, assistance with the research proposal, and conversations about design and architecture:

Nayda & Reza Boojar, Anthony Burke, Beverley Garlick, Maryam Gusheh, Martin Hagemann, Sandra Kaji-O'Grady, Simon Keizer, Adrian Lahoud, Maryam Majd, Salman Mohammadi, Kirsten Orr, Kahveh Rashidzadeh, and Sam Spurr.

Photographs from Iran can be viewed at the following web address:

www.flickr.com/photos/28055157@N00/sets/72157604524955827/

Quasiperiodic Tilings
Byera Hadley Travelling Scholarship Report
Andrew Willes, University of Technology, Sydney

Synopsis

There are a diverse variety of pentagonal and decagonal tiling patterns in medieval Islamic architecture. The remarkably similar properties of Timurid-period tilings and contemporary Penrose tilings suggests that quasiperiodic tilings were known to Persian mathematicians and architects five hundred years prior to their western counterparts. Quasiperiodic tilings display infinite variation and self-similarity, where similar patterns recur over multiple scales.

The first part of this report focuses on several tilings from the Friday Mosque in Esfahan, Iran, and confirms their compatibility with a recently posited theory that complex tilings with five-fold symmetry were constructed using various permutations of five template 'girih' tiles. The girih tile method alleviates the difficulties associated with traditional techniques of tile patterning using a ruler and compass, where slight errors in angle would propagate and be magnified over the thousands of lines that comprise the most elaborate tilings. The girih tile theory also naturally accounts for those tilings that exhibit two spatial tiling scales, where each large-scale girih tile can be neatly divided into a quasiperiodic arrangement of small-scale girih tiles.

These techniques resonate with contemporary architectural thought regarding algorithmic form generation methods and parametric systems. The parametric nature of the Timurid-period tilings is evident in the dual relationship between decoration and structure, where nodal points of the large-scale girih tiles constrain the arch geometry in a similar manner to the control points of spline curves.

Previously studied girih tilings exhibit either a periodic or quasiperiodic arrangement of girih tiles at the larger tiling scale. In contrast, two tilings analysed in this report display a less rigid adherence to the mathematical geometry, and reveal the role of a designer's discretion. One of these tilings, which appears to be a free-form arrangement of girih tiles, strengthens the girih tile theory because its simplicity makes alternative generation methods appear elaborately contrived. In the second example, the quasiperiodic symmetry is broken with the insertion of an irregular girih tile to optimise the compatibility between the building structure and decoration. Again, this interplay mirrors contemporary ideas on massaging the often messy synthesis between algorithmic and parametric form generation and real-world architecture.

The second part of this report explores the generative potential of quasiperiodic tilings in two and three dimensions in a contemporary architectural context, utilizing current mathematical knowledge of quasiperiodic tilings and digital techniques. Graphically intuitive algorithmic methods for generating Penrose tilings of arbitrary extent and their self-similar deflations are outlined, together with a technique to generate quasiperiodic girih tilings of arbitrary extent.